Two-Commodity Network Flow Formulations for Vehicle Routing Problems with Simultaneous Pickup & Delivery and a Many-to-Many Structure

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Abstract

Vehicle routing problems with pickup and delivery are of primary importance in distribution and logistics. Given a set of non-homogeneous vehicles, which are located at one central depot, the objective is to generate an optimal route plan ensuring that the demand for heterogeneous commodities can be satisfied by an arbitrary set of suppliers. Owing to their physical property, commodities have to be stored in either the same or separate load capacities during transportation (e.g., piece goods in contrast to liquids or bulk goods). Novel models are presented for these transport requirements generalizing existing models. The paper was motivated by a real-world application using the presented models for a combined approach to inventory management and vehicle routing. A case study validates the proposed methods and enriches the scope of application.

1 Introduction

Static vehicle routing problems (VRP) form an important class of pickup and delivery problems (PDP). Assume a symmetric road network is given as a simple undirected graph \(G = (V, E)\) with a vertex set \(V = \{0,1,\ldots,n+1\}\) that includes a set of customer vertices \(C = \{1,\ldots,n\}\). The depot is represented by a vertex \(\{0\}\) and an artificial vertex \(\{n+1\}\). The set of edges is given by \(E = \{(i,j) ; i,j \in V, i < j\} \setminus \{(0,n+1)\}\) and \(A = \{1,\ldots,m\}\) defines a set of heterogeneous commodities that cannot be substituted for each other. Let a subset \(D_a \subseteq C\) and the artificial vertex demand a quantity \(d_{ai} \geq 0 \ \forall i \in D_a \cup \{n+1\}\) of a commodity \(a \in A\) from a subset \(S_a \subseteq C\) and from the depot vertex. The quantity of supply is \(s_{ai} \geq 0 \ \forall i \in S_a \cup \{0\}\).

This situation reflects a many-to-many structure. Assume a balanced problem with each vertex either demanding or supplying commodities (i.e., \(\bigcup_{a \in A} D_a \cup S_a = C\), for all \(a \in A\) it holds \(s_{ai} \cdot d_{ai} = 0\), \(D_a \cap S_a = \emptyset\), and \(\sum_{i \in D_a \cup \{n+1\}} d_{ai} = \sum_{i \in S_a \cup \{0\}} s_{ai}\)). Customers that are

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not involved in allocation do not exist. Moreover, a set of \( M = \{1, \ldots, |M|\} \) delivery vehicles is located at the depot. In order to impose a maximum load capacity on the vehicle fleet, the physical and chemical properties of commodities have to be taken into account. For instance, liquids or bulk goods require different load capacities \( Q_{ak} \geq 0 \forall a \in A, k \in M \) if they are transported on the same vehicle, whereas piece goods can share a common capacity \( Q_k \geq 0 \forall k \in M \). To accomplish delivery, each vehicle leaves the depot \( \{0\} \), visits one or more customers in the sequence given by its assigned path, and returns to the artificial vertex \( \{n+1\} \). A simultaneous pickup and delivery operation is performed at each customer, and split delivery is not permitted. Consequently, the set of customers is partitioned into \(|M|\) Hamiltonian paths or vehicle routes, and the only intersections of all routes are the depot vertices. The edge weights \( c_{ij} \geq 0 \forall (i,j) \in E \) are measured in the shortest distances between two vertices. Depending on the planning situation, they can be associated with travel time, travel distance, monetary values (e.g., driver wages, costs for using toll roads, maintenance costs, implicit costs, internal prices), opportunity costs (e.g., staff is not available because of being en route), or can refer to other economic aspects. Note that \( c_{i,n+1} = c_{0i} \forall i \in C \). The objective is to design a collection of vehicle routes with the following properties: all vertices are visited exactly once; all vehicles are en route; the capacity of each vehicle is never exceeded by the vehicle load; and the total travel costs are minimized.

Dell’Amico et al. (2006) propose a branch-and-price approach for the NP-hard vehicle routing problem with simultaneous pickup and delivery (VRP-SPD). This problem often occurs in reverse logistics. It has a one-to-many-to-one structure with identical vehicles and is a two-commodity model in which the first (second) commodity is supplied from the depot (the customers) to the customers (the depot) (i.e., \( A = \{1,2\} \), \( O_{1k} = O_{2k} = Q \forall k \in M \), \( D_1 = C \), \( s_{10} = \sum_{i \in D_1} d_{1i} \), \( d_1,n+1 = 0 \), \( S_2 = C \), \( s_2,0 = 0 \), \( d_2,n+1 = \sum_{i \in S_2} s_{2i} \)). Hernández-Pérez and Salazar-González (2004) examine the one-commodity pickup and delivery traveling salesman problem (1-PDTSP), which has a many-to-many structure, one vehicle, and no capacity restrictions (i.e., \( A = \{1\} \), \( M = \{1\} \), and \( Q = \infty \)). So far, multiple vehicle problems with a many-to-many structure have received less attention. One approach, for example, is the express delivery problem (EDP) in which pickup and delivery operations are performed separately on different distribution levels (Galvão 2007). Savelsbergh and Sol (1995) survey PDPs and VRPs using a generalized PDP. Berbeglia et al. (2007) provide a classification scheme for PDPs, which is used in this paper. The VRP is closely related to the capacitated vehicle routing problem (CVRP), which is a single commodity problem in which customers are served from the depot (i.e., \( A = \{1\} \), \( D = C \), \( s_0 = \sum_{i \in D} d_i \), and \( d_{n+1} = 0 \)). Setting the capacity sufficiently high
with $Q_k = \infty \ \forall k \in M$, the CVRP is equivalent to the multiple traveling salesman problem (MTSP). A given CVRP instance has a feasible solution if and only if there exists a feasible solution to the corresponding bin-packing problem (BPP) that is obtained with $c_{ij} = 0 \ \forall [i, j] \in E$ (Ralphs et al. (2003)). Exact methods are surveyed in Laporte (1992) and Fisher (1995). Toth and Vigo (2002) give a general overview of CVRPs. Mingozi et al. (2007) report recent advances. Baldacci et al. (2004) present a two-commodity network flow formulation that provides the strongest lower bound so far. Desrochers et al. (1990) give a classification scheme for VRPs.

The aim of this paper is to provide tight mathematical models of practical relevance. First, it is assumed that the vehicle load is described by a linear function. Exact values of the vehicle load are obtained for liquids and bulk goods since they are divisible. The vehicle load for piece goods, however, is approximated because an exact determination means solving a two or three-dimensional BPP simultaneously to the VRP. Second, the graph $G$ is undirected leading to an approximation of the path length for intra-city traffic if one-way streets occur. However, as distances between cities are often symmetric, the modeling approach defines one flow path for the vehicle load and another for the empty space. Both assumptions have an important impact on the complexity and drastically reduce the number of integer variables. This allows small-sized real-world instances to be solved within reasonable time. Thus, the presented models can be used as a decision support. The paper is organized as follows: section 2.1 presents a one-commodity model of the VRP with pickup and delivery based on a two-commodity network flow formulation for a uniform vehicle fleet. Subsequently, the approach is extended to multiple commodities and non-homogeneous vehicles. A model for commodities sharing one capacity during transportation is given in §2.2. Separate capacities are considered in §2.3. A case study of a company trading with tires is provided in §3, which presents an integrated approach to inventory management between branches and vehicle routing. Conclusions are given in §4, and proofs are given in the appendix.

2 Mathematical Formulations

2.1 Single Commodity

Consider a single commodity $A = \{1\}$ and a fleet of identical vehicles $Q_k = Q \ \forall k \in M$. Assume w.l.o.g. $d_i > 0 \ \forall i \in D$ and $s_i > 0 \ \forall i \in S$. In the following two-commodity network flow formulation, two flow paths are defined for each route. Starting from the depot vertex $[0]$ and ending at the artificial vertex $[n+1]$, the first flow path represents the vehicle load. The second flow path, which defines the empty space on the vehicle, has the opposite direction. Let $x_{ij}, x_{ji} \geq 0 \ \forall [i, j] \in E$ be continuous flow variables representing these flow paths. If an edge is selected in a solution, the binary variable $y_{ij} \in \{0, 1\} \ \forall [i, j] \in E$ equals 1 ($y_{ij} = 0$, other-
wise). Assume a vehicle travels from vertex $i$ to $j$. Then the flow variable $x_{ij}$ represents the load on the vehicle, and the empty space on the vehicle is given by $x_{ji} = Q - x_{ij}$. According to Berbeglia et al. (2007), the problem is denoted as the One-Commodity VRP with Pickup and Delivery [M-M|P/D|m]. Its formulation is:

$$\min \sum_{[i,j] \in E} c_{ij}y_{ij}$$  \hspace{1cm} (1)

subject to

$$\sum_{j \in V\setminus i} (x_{ji} - x_{ij}) = \begin{cases} -2s_i, & \forall i \in S \\ 2d_i, & \forall i \in D \end{cases}$$ \hspace{1cm} (2)

$$x_{ij} + x_{ji} = Qy_{ij} \hspace{1cm} \forall [i,j] \in E$$ \hspace{1cm} (3)

$$\sum_{j \in C} x_{0j} = |M|Q - s_0$$ \hspace{1cm} (4)

$$\sum_{j \in C} x_{n+1j} = |M|Q - d_{n+1}$$ \hspace{1cm} (5)

$$\sum_{j \in C} x_{0j} = s_0$$ \hspace{1cm} (6)

$$\sum_{j \in C} x_{nj+1} = d_{n+1}$$ \hspace{1cm} (7)

$$\sum_{i < j} y_{ij} + \sum_{j \in V\setminus i} y_{ji} = 2 \hspace{1cm} \forall i \in C$$ \hspace{1cm} (8)

$$x_{ij}, x_{ji} \geq 0 \hspace{1cm} \forall [i,j] \in E$$ \hspace{1cm} (9)

$$y_{ij} \in \{0,1\} \hspace{1cm} \forall [i,j] \in E$$ \hspace{1cm} (10)

The objective function (1) minimizes the total distribution costs. (2) defines flow conservation for the customer vertices. Although the flow variables are continuous, their values are integer if all quantities $d_i \in \mathbb{N} \forall i \in D \cup \{n+1\}$ and $s_i \in \mathbb{N} \forall i \in S \cup \{0\}$ are also integers. The capacity constraint (3) implies that the flows of load and of empty space are indistinguishable because it is defined for all edges $y_{ij}$. Note that this disallows the implementation of time windows and asymmetric problems. The empty space (load) on the vehicle fleet leaving the depot is defined by (4) ((6)), and the empty space (load) arriving at the artificial vertex is given by (5) ((7)). Equation (8) defines a vertex degree of two for each customer vertex. This implies that all
customer vertices are visited once, either to load $s_i$ or to unload $d_j$. The domains of the decision variables are defined by (9) and (10).

**Remark 1:** The modeling approach can also be applied to asymmetric problems. Then both flow paths cannot be expressed by identical variables. Individual variables for vehicle load and for empty space have to be introduced. However, the empty space variables are redundant.

**Example 1:** Suppliers are $S = \{1, 3\}$ with $s_1 = 4$ and $s_3 = 1$. Customers are given by $D = \{2, 4\}$ with $d_2 = d_4 = 2$, $d_{n+1} = 1$. One delivery vehicle (i.e., $M = \{1\}$) has a capacity of $Q = 5$, and edge costs are $c_{ij} = 1 \forall [i, j] \in E$. Figure 1 shows a feasible solution with total costs of five monetary units, and the route is $[0, 1, 4, 3, 2, n+1]$. The diagram shows the values of the flow variables. The flow path $p^L$ defines the vehicle load and is illustrated on axis $x_{ij}$ and by the dotted area. $P^{ES}$ on axis $x_{ji}$ depicts the empty space. The example is equivalent to the 1-PDTSP.

![Illustration of a feasible solution to Example 1](image)

**Figure 1:** Illustration of a feasible solution to Example 1

**Lemma 1:** Each feasible solution to the model (1)-(10) is a VRP with Pickup and Delivery because it has the following properties:

a) All vehicles leave the depot and arrive at the artificial vertex.

b) Each vehicle route is a Hamiltonian path, and the capacity is never exceeded.

c) It holds for each vehicle route that a vehicle must pick up a commodity before delivery is possible and that all pickup and delivery requests are satisfied.

**Proof:** See the appendix.
2.2 Multiple Commodities and Single Capacity

In this model, it is assumed that all commodities share the same vehicle capacity during transportation. Moreover, all vehicles are non-homogeneous (i.e., $Q_k \geq 0 \ \forall k \in M$). Situations of unbalanced demand and supply are also implemented. It implies that vertices not involved in allocation do not need to be visited. This decision becomes part of the planning problem. Therefore, a solution has $|M|$ Hamiltonian paths for a subset of customer vertices. Define for each commodity $a \in A$ a binary parameter $w_a$ that takes the value 1 if the cumulated demand exceeds the cumulated supply with $\sum_{i \in D_a \cup \{n+1\}} d_{ai} > \sum_{i \in S_a \cup \{0\}} s_{ai}$. Then variable $\delta_{ai}$ represents the unsatisfied demand of a commodity $a \in A$ by vertex $i \in D_a \cup \{n+1\}$, and $d_{ai} - \delta_{ai} \geq 0$ defines the net demand. Otherwise (i.e., $w_a = 0$), variable $\sigma_{ai}$ represents the remaining supply of a commodity $a \in A$ by vertex $i \in S_a \cup \{0\}$, and the net supply is given by $s_{ai} - \sigma_{ai} \geq 0$. Note that variable $\sigma_{ai}$ (parameter $s_{ai}$) can be interpreted as the stock of a commodity after (before) the vehicle has visited that vertex. Let the binary variable $z_{ik}$ take the value 1 if a customer vertex $i \in C$ is traversed by a vehicle $k \in M$ ($z_{ik} = 0$, otherwise). Define $x_{aij}, x_{aji} \geq 0 \ \forall a \in A \cup \{m+1\}$ and $y_{ijk} \in \{0,1\} \ \forall k \in M$ for all edges $[i, j] \in E$. If an edge is traversed by a vehicle, then $y_{ijk}$ equals 1 ($y_{ijk} = 0$, otherwise). In addition, the flow variable $x_{aij}$ represents the load of a commodity on a vehicle that travels from vertex $i$ to $j$, and the empty space is given by $x_{m+1 ji} = Q_k - \sum_{a \in A} x_{aij}$. This problem can be classified as the VRP with Simultaneous Pickup and Delivery for Multiple Commodities sharing One Capacity [M-M|PD|m]. The formulation is as follows:

$$\min \sum_{(i,j) \in E} \sum_{k \in M} c_{ij}y_{ijk} \quad (11)$$

subject to

$$\sum_{a \in A \cup \{m+1\}} \sum_{j \neq i} x_{aij} = \sum_{a \in A} 2(d_{ai} - \delta_{ai}) - \sum_{a \in A} 2(s_{ai} - \sigma_{ai}) \quad \forall i \in C \quad (12)$$

$$\sum_{j \neq i} \sum_{a \in A} \left( x_{aij} - x_{aji} \right) = \begin{cases} -s_{ai} + \sigma_{ai} & \forall a \in A, i \in S_a \\ d_{ai} - \delta_{ai} & \forall a \in A, i \in D_a \end{cases} \quad (13)$$
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\[
\sum_{a \in A \cup \{m+1\}} (x_{aij} + x_{aji}) = \sum_{k \in M} Q_k y_{ijk} \quad \forall [i, j] \in E \tag{14}
\]

\[
\sum_{j \in C} x_{m+1} j_0 = \sum_{k \in M} Q_k - \sum_{a \in A} (s_{a0} - \sigma_{a0}) \tag{15}
\]

\[
\sum_{j \in C} x_{m+1} n+1 j = \sum_{k \in M} Q_k - \sum_{a \in A} (d_{a n+1} - \delta_{a n+1}) \tag{16}
\]

\[
\sum_{j \in C} x_{a0} j = s_{a0} - \sigma_{a0} \quad \forall a \in A \tag{17}
\]

\[
\sum_{j \in C} x_{aj} n+1 = d_{a n+1} - \delta_{a n+1} \quad \forall a \in A \tag{18}
\]

\[
\sum_{i \in D_a \cup \{n+1\}} \delta_{ai} = w_a \left( \sum_{i \in D_a \cup \{n+1\}} d_{ai} - \sum_{i \in S_a \cup \{0\}} s_{ai} \right) \quad \forall a \in A \tag{19}
\]

\[
\sum_{i \in S_a \cup \{0\}} \sigma_{ai} = (1 - w_a) \left( \sum_{i \in S_a \cup \{0\}} s_{ai} - \sum_{i \in D_a \cup \{n+1\}} d_{ai} \right) \quad \forall a \in A \tag{20}
\]

\[
\delta_{ai} \leq d_{ai} \quad \forall a \in A, i \in D_a \cup \{n+1\} \tag{21}
\]

\[
\sigma_{ai} \leq s_{ai} \quad \forall a \in A, i \in S_a \cup \{0\} \tag{22}
\]

\[
\sum_{k \in M} z_{ik} \leq 1 \quad \forall i \in C \tag{23}
\]

\[
\sum_{j \in C} y_{00} j = 1 \quad \forall k \in M \tag{24}
\]

\[
\sum_{j \in V} y_{ijk} + \sum_{j \in V} y_{jik} = 2z_{ik} \quad \forall i \in C, k \in M \tag{25}
\]

\[
z_{ik} \in \{0, 1\} \quad \forall i \in C, k \in M \tag{26}
\]

\[
y_{ijk} \in \{0, 1\} \quad \forall [i, j] \in E, k \in M \tag{27}
\]

\[
\delta_{ai} \geq 0 \quad \forall a \in A, i \in D_a \cup \{n+1\} \tag{28}
\]

\[
\sigma_{ai} \geq 0 \quad \forall a \in A, i \in S_a \cup \{0\} \tag{29}
\]

\[
x_{aij} \geq 0 \forall a \in A \cup \{m+1\}, [i, j] \in E \tag{30}
\]

The objective function is given by (11). Equation (12) gives the flow pattern of the empty space variables \(x_{m+1} j_0\) and \(x_{m+1} ji\) based on the load of all commodities defined by (13). In (13), net flow patterns of the flow variables \(x_{aij}\) and \(x_{aji}\) are defined separately for each
commodity that is allocated from \(S_a\) to \(D_a\). Note that the constraints \(\sum_{j \in V} (x_{aji} - x_{aij}) = 0\) \(\forall a \in A, i \in C \setminus (S_a \cup D_a)\) are redundant. The individual capacity \(Q_k\) of each vehicle is taken into account by capacity constraint (14). Because of (23) and (25), only one parameter \(Q_k\) is set on the right-hand side of (14). The net empty space (net load of a commodity) is defined for the depot by (15) ((17)) and for the artificial vertex by (16) ((18)). If the cumulated demand exceeds the cumulated supply of a commodity (i.e., \(w_a = 1\)), (19) defines the surplus of demand as \(\sum_{i \in S_a \cup \{0\}} \delta_{ai}\). Otherwise (i.e., \(w_a = 0\)), the cumulated remaining supply is given as \(\sum_{i \in S_a \cup \{0\}} \sigma_{ai}\) in (20). Inequality (21) ((22)) prevents the unsatisfied demand \(\delta_{ai}\) (remaining supply \(\sigma_{ai}\)) from being numerically larger than the demand \(d_{ai}\) (supply \(s_{ai}\)). Note that a demand vertex \(i \in D_a \cup \{n+1\}\) (supply vertex \(i \in S_a \cup \{0\}\)) would supply (demand) a commodity if \(\sum_{j \in V} x_{aji} < \sum_{j \in V} x_{aij}\) \(\sum_{i \neq j} x_{aji} > \sum_{i \neq j} x_{aij}\) was allowed in (13). A customer vertex that is not involved in allocation can be omitted in a solution (i.e., \(\sum_{k \in M} z_{ik} = 0\) in (23) if \(s_{ai} - \sigma_{ai} = 0\) for \(i \in S_a\) or \(d_{ai} - \delta_{ai} = 0\) for \(i \in D_a\) holds for all \(a \in A\)). Applying vertex coloring, a customer vertex that is included in a solution can only be visited by one vehicle (i.e., \(\sum_{k \in M} z_{ik} = 1\)). Constraint (24) states that every vehicle has to leave the depot. Relaxing (24) may lead to an assignment of two or more routes to one vehicle. Note that \(\sum_{j \in C} y_{j n+1} = 1 \forall k \in M\) is superfluous. Equation (25) defines a feasible vertex degree of either zero or two for any vertex traversed by a vehicle. Because of (23) and (25), each edge is assigned to at most one vehicle. (30)-(27) define variable domains.

**Remark 2:** A further reduction of binary variables is possible by replacing (26) with \(z_{ik} \geq 0 \forall i \in C, k \in M\) and by adding the constraints \(y_{ijk} \leq z_{ik} \forall i \in C, j \in V, k \in M, i < j\) and \(y_{ijk} \leq z_{jk} \forall i \in V, j \in C, k \in M, i < j\) to the formulation above.

**Example 2:** Assume two commodities \(A = \{1, 2\}\). The supply for the first commodity is ensured by \(S_1 = \{1, 3, 5\}\), and customers are represented by \(D_1 = \{2, 4\}\). It is \(s_{15} = 1\), \(d_{1n+1} = 2\), and additional data is given in Example 1. The second commodity has \(S_2 = \{0, 4\}\) with \(s_{20} = s_{24} = 2\), and \(D_2 = \{2, 3, 6\}\) with \(d_{22} = d_{23} = 1\), \(d_{25} = 6\). Two vehicles (i.e., \(M = \{1, 2\}\)) are available with a capacity of \(Q_1 = Q_2 = 4\), and edge costs are defined as \(c_{ij} = 1 \forall [i, j] \in E\). Figure 2 shows a feasible solution with total costs of seven monetary units.
Dotted (hatched) areas represent the first (second) commodity. The surplus of demand is $\delta_{25} = 4$ units of the second commodity at vertex $\{5\}$.

**Figure 2:** Illustration of a feasible solution to Example 2

**Lemma 2:** Given a feasible solution to the model (11)-(30) that includes a customer vertex $i \in C$ and an edge $[i, h] \in E$. If a vehicle $k \in M$ is loaded with $\sum_{a \in A} x_{ah} \geq 0$, then (12) defines the empty space on the vehicle as $x_{m+1} = Q_k - \sum_{a \in A} x_{ah}$.

**Proof:** See the appendix.

### 2.3 Multiple Commodities with Separate Capacities

Assume a non-homogeneous vehicle fleet. Additionally, each commodity has to be stored separately during transportation (i.e., $Q_{ak} \geq 0 \forall a \in A, k \in M$). Define variables $x_{aij}, x_{aji} \geq 0 \forall a \in A$ and $y_{ijk} \in \{0,1\} \forall k \in M$ for all edges $[i, j] \in E$. If an edge is traversed by a vehicle, the binary variable $y_{ijk}$ takes the value 1 ($y_{ijk} = 0$, otherwise). If a vehicle travels from vertex $i$ to $j$, the flow variable $x_{aij}$ represents the load of a commodity, and $x_{aji} = Q_{ak} - x_{aij}$ defines the empty space. Parameters and additional variables are defined in accordance with §2.2. The problem can be classified as the VRP with Simultaneous Pickup and Delivery for Multiple Commodities with Separate Capacities [M-M|PD|m]. Its formulation is:

$$\min \sum_{[i,j] \in E} \sum_{k \in M} c_{ij} y_{ijk}$$

subject to (17)-(29) and

$$\sum_{j \in V \atop i \neq j} (x_{aij} - x_{aji}) = \begin{cases} -2(s_{ai} - \sigma_{ai}), & \forall a \in A, i \in S_a, \\ 2(d_{ai} - \delta_{ai}), & \forall a \in A, i \in D_a \end{cases}$$

(31)
\[ x_{aij} + x_{aji} = \sum_{k \in M} Q_{ak} V_{ijk} \quad \forall a \in A, [i, j] \in E \]  
(32)

\[ \sum_{j \in C} x_{aj0} = \sum_{k \in M} Q_{ak} - s_{a0} + \sigma_{a0} \quad \forall a \in A \]  
(33)

\[ \sum_{j \in C} x_{a_{n+1} j} = \sum_{k \in M} Q_{ak} - d_{a_{n+1}} + \delta_{a_{n+1}} \quad \forall a \in A \]  
(34)

\[ x_{aij}, x_{aji} \geq 0 \quad \forall a \in A, [i, j] \in E \]  
(35)

The objective function (11) and the constraints (17)-(29) are discussed in §2.2. The flow conservation constraints (31) are similar to (2) but use net values instead and are defined separately for each commodity. Constraints \( \sum_{j \in V} (x_{aji} - x_{aij}) = 0 \) \( \forall a \in A, i \in C \setminus (S_a \cup D_a) \) are redundant. The capacity constraint (32) is given for each commodity. Equation (33) ((34)) defines the net empty space of the vehicle fleet leaving the depot (arriving at the artificial vertex). Variable domains are defined by (35).

**Example 3:** Assume a vehicle transporting two dangerous goods in separate tanks, \( A = \{1, 2\} \).

Let \( S_1 = D_2 = \{1, 2, \ldots, 5\} \). Two tankers (i.e., \( M = \{1, 2\} \)) collect quantities of waste oil \( s_{ai} \forall i \in S_1 \) from gas stations, and transport them to the depot (i.e., \( d_{1_{n+1}} = B \) with \( B \) as a big number). The gas stations demand gasoline \( d_{2i} \forall i \in D_2 \), which is not a scarce good (i.e., \( s_{20} = B \)). Quantities and capacity equipments are shown in Table 1, and Figure 3 illustrates a feasible solution. All quantities are given in liters. The upper (lower) diagrams refer to waste oil (gasoline). Despite separate capacities, this example is an analogon to the VRP-SPD.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>( Q_{a1} )</th>
<th>( Q_{a2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>waste oil ( s_{ai} ) ( a = 1 )</td>
<td>2000</td>
<td>1600</td>
<td>1000</td>
<td>650</td>
<td>2300</td>
<td>5000</td>
<td>3000</td>
</tr>
<tr>
<td>gasoline ( d_{ai} ) ( a = 2 )</td>
<td>8000</td>
<td>4000</td>
<td>5000</td>
<td>6000</td>
<td>8000</td>
<td>20000</td>
<td>20000</td>
</tr>
</tbody>
</table>
A Case Study of an Application

Consider a company trading with car tires that is subdivided into eight branches. Taking advantage of a bonus on freight costs for any order on tires from a wholesaler, one central depot with delivery vehicles is established. One branch is located at the depot \((n = 7)\). Tire allocation can be considered as a problem with two distribution levels. A customer buys tires at a branch, and the branches demand tires from the depot considering a reorder point. The service goal of the company is to satisfy customer demands within the shortest time possible. Long travel times imply high opportunity costs because a worker is en route instead of assembling tires. The objective is to minimize the distribution costs resulting from transports between the depot and the branches.

One major improvement is that requests can be satisfied by the depot as well as by any other branch. Thus, the stock turnover rate is increased, and stocks are balanced. This is of advantage because expenses for capital commitment caused by stocks can be lowered. Moreover, a higher service level is achieved on the long run because the availability of commodities is higher. This integrated approach significantly improves the quality of the supply chain. Hierarchical goal systems are often adopted in order to reflect conflicting objectives as well as to reduce the complexity of the business process. Therefore, vehicle routing is given a higher priority than
crew scheduling. An appropriate vehicle routing solution to crew scheduling, which is obtained from the model (11)-(30), is outlined in the following qualitative discussion.

Two delivery vehicles with different capacities are available ($M = \{1, 2\}$). Assuming that tires have approximately the same size and volume, the bin packing problem for the vehicles is of less importance, and the capacity can be approximated with

$$Q_k = \frac{\text{Volume}_{k-th \text{ vehicle}}}{\text{Volume}_{\text{tire}}}.$$  

Because a vehicle can be scheduled for several routes in succession, the instances were solved using all possible vehicle combinations, and a decision maker was provided with a set of solutions. Moving stock to the depot creates inventory capacity and reflects seasonal changes of supply ($A = \{1, ..., 19\}$). It was suggested to minimize the total travel time instead of distribution costs because of four reasons. First, travel costs are often a non-decreasing function of the travel time. Second, labor costs are much higher than costs related to vehicles. Third, it was not possible to evaluate opportunity costs for the crew. Finally, lowering the travel time implies a better service level. Travel times were obtained by measuring standardized routes and by calculating the average. The calculations of the model (11)-(30) were performed on an Intel Pentium IV 3.00 GHz equipped with a Linux 2.6.8-2-386 kernel, ILOG AMPL 8.000, CPLEX 8.000, and 2 GB memory. Figure 4 depicts the optimal solutions for a representative data set and gives the duration of the routes. Arcs represent the sequence of the branches in the routes. The demand of each customer depends on a discrete identical independent distribution function, and at most ten commodities per branch are requested as well as supplied. The computational times were 3.0 sec (0.2 sec, 29.4 sec) for the first (second, third) instance. Typical instances of small-sized companies that are specialized in trade and not in transportation are slightly more complex. In forwarding agencies, there are usually more than one hundred vehicles and customers.

![Figure 4: Optimal routes of the case study](image)

**Figure 4:** Optimal routes of the case study
A dispatcher chooses an appropriate solution to crew scheduling by considering the travel time of each route, the working activity, and the number of available workers. General conclusions based on the solution characteristics were identified. Since at least one customer is assigned to each vehicle, solutions with a suboptimal number of vehicles are likely to appear. The first route plan on the left in Figure 4 is a typical solution to this situation. The second route plan is more preferable than the first one because it includes all branches, requires only one vehicle, and the travel time is only slightly larger. Often the demand distribution function cannot be estimated properly. Peaks of unexpectedly high working loads can occur, which leads to a bottleneck of available capacity and conflicts with the service goal. It is suggested to apply solutions with long travel times on days where only a low amplitude of the demand is expected (i.e., Monday morning or Saturday). In general, solutions with a structure similar to the third route plan are more advantageous. Note that demands of single-customer routes that are not urgent can be satisfied later within a rolling planning horizon.

4 Conclusions

This paper considers static vehicle routing problems (VRP) with pickup and delivery. In a many-to-many structure, it is assumed that requests on commodities can be satisfied by an arbitrary set of suppliers. To accomplish allocation, the vehicle fleet is located at one central depot. The objective is to find a Hamiltonian circuit for each vehicle with minimal total travel costs such that vehicle capacity is never exceeded. Using the notation provided by Berbeglia et al. (2007), a formulation for the One-Commodity VRP with Pickup and Delivery [M-M|P/D|m] is provided for a uniform vehicle fleet. Moreover, the physical property of commodities is an important aspect in transportation. Heterogeneous commodities can either share the same load compartments or have to be stored separately (i.e., piece goods in contrast to liquids or bulk goods). Considering their transport requirements, two novel models for the VRP with Simultaneous Pickup and Delivery for Multiple Commodities sharing One Capacity and with Separate Capacities [M-M|PD|m] are presented for a non-homogeneous vehicle fleet. An integrated approach to an inventory management and vehicle routing gives new impetus to the scope of application for multiple vehicle problems. The approach is discussed in a case study. Providing tight formulations with few binary variables, small real-world instances were solved to optimality by standard solvers within reasonable time. Further research recommends a real-time implementation since static planning is not capable of handling short term changes, such as new requests, vehicle breakdowns, or disturbances of the transportation network.

5 Appendix

Proof of Lemma 1: a) Adding (4) and (6) ((5) and (7)) as well as using (3), the vertex degree of the depot \( \{0\} \) (artificial vertex \( \{n+1\} \)) is \( |M| \). Thus, a) holds.
b) (8) ensures that each path from \([0]\) to \([n+1]\), which is induced by the edge set \([\{i,j\} \in E \mid y_{ij} = 1]\), includes vertices with a degree of two. Owing to (3), paths assigned to the flow variables \(x_{ij}\) and \(x_{ji}\) are pairwise disjoint (\(y_{ij} = x_{ij} = x_{ji} = 0\), otherwise). Add equation (2) for all vertices that are included in the set \(P \subseteq C\) with \(|P| \geq 2\) in order to obtain:

\[
\sum_{i \in P \backslash D} 2d_i - \sum_{i \in P \backslash S} 2s_i = \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (x_{ji} - x_{ij}) = \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (x_{ji} - x_{ij}) + \sum_{i \in P} \sum_{j \in P, i \neq j} (x_{ji} - x_{ij})
\]

\[
= \sum_{i \in P, j \in V \backslash P} (x_{ji} - x_{ij}) + \sum_{i \in P, j \in P, i \neq j} (x_{ji} - x_{ij}) + \sum_{i \in P, j \in P, i \neq j} (x_{ji} - x_{ij})
\]

\[
= \sum_{i \in P, j \in V \backslash P} (x_{ji} - x_{ij})
\]

\(\forall P \subseteq C, |P| \geq 2\)

To get two inequalities, apply \(-x_{ij} \leq 0\), \(x_{ji} \geq 0\), and \(x_{ij} > 0 \lor x_{ji} > 0 \Rightarrow y_{ij} = 1\) from (3).

\[
\sum_{i \in P \backslash D} 2d_i - \sum_{i \in P \backslash S} 2s_i = \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (x_{ji} - x_{ij}) \leq \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} x_{ji} \leq Q \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (y_{ji} + y_{ij})
\]

\[
\geq -\sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} x_{ij} \geq -Q \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (y_{ji} + y_{ij})
\]

Multiplying the second inequality with \(-1\) and taking the maximum of the two left-hand sides,

\[
2 \left| \sum_{i \in P \backslash D} d_i - \sum_{i \in P \backslash S} s_i \right| \leq Q \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (y_{ji} + y_{ij})
\]

\(\forall P \subseteq C, |P| \geq 2\)

is valid. The inequality is bounded by the cardinality of the resulting cut:

\[
2 \left| \sum_{i \in P \backslash D} d_i - \sum_{i \in P \backslash S} s_i \right| Q^{-1} \leq \sum_{i \in P} \sum_{j \in V \backslash P, i \neq j} (y_{ji} + y_{ij})
\]

\(\forall P \subseteq C, |P| \geq 2\)

This leads to the well-known subtour elimination constraint that is presented in Nobert (1982), Laporte and Nobert (1983). The number of vehicles visiting and leaving \(P\) equals at least the number of vehicles that is necessary to satisfy the pickup and delivery requests of all customers in \(P\). The left-hand side is twice the trivial lower bound of the BPP. Thus, b) holds.
c) Let \( H = \{ i \in C \mid y_{ji} = 1 \land y_{ji} = 1, j \in V \} = \{ i_1, \ldots, i_{|H|} \} \) denote the set of vertices included in the Hamiltonian path \([i_0, \ldots, i_v, \ldots, i_{|H|+1}]\) of one vehicle that leads from the depot \( i_0 = 0 \) to the artificial vertex \( i_{|H|+1} = n+1 \). Let w.l.o.g. \( s_i = 0 \ \forall i \in D \cup \{ n+1 \} \), \( d_i = 0 \ \forall i \in S \cup \{ 0 \} \), and let \( [j,i],[i,h] \in E \) be two edges incident to a vertex \( i \in H \). Because (8) defines a vertex degree of two, equation (2) becomes:

\[
x_{ji} - x_{ij} + x_{hi} - x_{ih} = \begin{cases} 
-2s_i & \forall i \in S \cap H \\
2d_i & \forall i \in D \cap H 
\end{cases}
\]

Substitute \( x_{ji}, x_{hi} \) with (3), let \( j = i_{v-1}, i = i_v, h = i_{v+1} \), and apply the non-negativity constraint (9). The resulting transportation equation is illustrated in Figure 5.

\[
x_{i_v-i_{v-1}} = x_{i_v-i_{v-1}} + s_i - d_i \geq 0 \\
x_{i_{v+1}-i_v} = x_{i_{v+1}-i_v} = Q - x_{i_v-i_{v-1}}
\]

Figure 5: Illustration of the transportation equation

Using \( x_0 = \alpha \cdot s_0 \ \forall \alpha \in [0,1] \) from (6) and summing all vertices on the Hamiltonian path up to vertex \( i_v \), the following holds for the flow path that is assigned to the vehicle load:

\[
x_{0-i_1} = \alpha \cdot s_0 \geq 0 \\
x_{1-i_2} = x_{0-i_1} + s_{i_1} - d_{i_1} = \alpha \cdot s_0 + s_{i_1} - d_{i_1} \geq 0 \\
\ldots \\
x_{v-i_{v+1}} = \alpha \cdot s_0 + \sum_{i \in \{0, \ldots, i_v\}} (s_i - d_i) \geq 0 \ \forall i_v \in H
\]

The non-negativity constraint (9) implies that the difference is positive. Considering \( x_{i_{|H|+1}} = \beta \cdot d_{n+1} \ \forall \beta \in [0,1] \) from (7), each vehicle picks up commodities before delivery. Moreover, all pickup and delivery requests of the vertices on path \([i_0, \ldots, i_v] \forall i_v \in H \cup \{ n+1 \} \) are satisfied because of (2). Thus, c) holds.
Proof of Lemma 2: Assume a balanced problem w.l.o.g. The assumption implies \( y_{ihk} = z_{ik} = 1 \). Let \( y_{jik} = 1 \). (25) defines a vertex degree of two for \( y_{jik} + y_{ihk} = 2 \forall i \in C \).

(12) becomes:

\[
\sum_{a \in A \cup [m+1]} (x_{aji} - x_{ah}\text{,} i - x_{aih}) = \sum_{a \in A \setminus D_a} 2d_{ai} - \sum_{a \in A \setminus S_a} 2x_{ai} \quad \forall i \in C
\]

A substitution of \( x_{aji} \) and \( x_{aih} \) with (14) gives:

\[
\sum_{a \in A \cup [m+1]} (x_{aji} + x_{ah}) = Q_k + \sum_{a \in A \setminus D_a} d_{ai} - \sum_{a \in A \setminus S_a} s_{ai} \quad \forall i \in C
\]

(36)

Summing up (13) for all commodities \( a \in A \) allocated on the flow path \( x_{aji} \), \( x_{aih} \) and considering (25), it follows:

\[
\sum_{a \in A} x_{aji} = \sum_{a \in A} x_{aih} - \sum_{a \in A \setminus D_a} d_{ai} + \sum_{a \in A \setminus S_a} s_{ai} \quad \forall i \in C
\]

(37)

From (36) and (37)

\[
\sum_{a \in A} x_{aih} + \sum_{a \in A} x_{aih} + x_{m+1} ji + x_{m+1} hi = Q_k \quad \forall i \in C
\]

is obtained. Considering (13) with \( \sum_{a \in A} x_{aih} \geq 0 \), it follows \( \sum_{a \in A} x_{aih} = 0 \). To satisfy (14), \( x_{m+1} ji = 0 \) . Thus, the lemma holds. Figure 6 illustrates the proof.

Figure 6: Illustration of Lemma 2
References


