4 Hitchcock Transportation Problem

The balanced transportation problem is defined as follows:

$c_{i,j}$: Delivery costs for each product unit that is transported from supplier i to customer j

a_i : Total supply of $i = 1,...,m$

b_j : Total demand of $j = 1,...,n$

$x_{i,j}$: Quantity that supplier $i = 1,...,m$ delivers to the customer $j = 1,...,n$

$$(P) \text{Minimize } c^T \cdot x$$

$$\text{s.t.} \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \ldots & \ldots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} \cdot x = \begin{pmatrix} a_1 \\ \ldots \\ \ldots \\ a_m \\ b \end{pmatrix}$$

$$x = \left(x_{1,1},...,x_{1,j},...,x_{1,n},...,x_{i,1},...,x_{i,n}...,x_{m,1},...,x_{m,n}\right)^T \geq 0$$

The dual problem

- Thus, we obtain as the dual problem

$$(D)\ \text{Maximize} \sum_{i=1}^{m} a_i \cdot \pi_i + \sum_{j=1}^{n} b_j \cdot \pi_{m+j} = \sum_{i=1}^{m} a_i \cdot \alpha_i + \sum_{j=1}^{n} b_j \cdot \beta_j \ \text{ s.t.}$$

$$\begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \ldots & & E_n \\ & & \ldots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \pi \le \begin{pmatrix} c_{1,1} \\ \ldots \\ c_{i,1} \\ \ldots \\ c_{m,n} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \ldots & & E_n \\ & & \ldots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \le \begin{pmatrix} c_{1,1} \\ \ldots \\ c_{i,1} \\ \ldots \\ c_{m,n} \end{pmatrix}$$

i.e.

$$\forall i \in \{1,\ldots,m\} : \forall j \in \{1,\ldots,n\} : \alpha_i + \beta_j \le c_{i,j}$$

$$\forall i \in \{1,\ldots,m\} : \alpha_i \text{ free} \wedge \forall j \in \{1,\ldots,n\} : \beta_j \text{ free}$$

4.1 Using the Simplex Algorithm

- Relevant costs are calculated as follows

$$\forall i \in \{1,\ldots,m\} : \forall j \in \{1,\ldots,n\} : \overline{c}_{i,j} = c_{i,j} - \left(\pi^T \cdot A\right)_{(i-1)\cdot n+j} = c_{i,j} - \left(A^T \cdot \pi\right)_{(i-1)\cdot n+j}$$
$$= c_{i,j} - \alpha_i - \beta_j$$

- Observation: Consider the matrix *A*

$$A = \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \ldots & \ldots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix}$$

Transportation matrix

$$A = \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \dots & \dots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \\ a_m \\ \hat{A} \end{pmatrix} = \begin{pmatrix} a_1 \\ \\ a_m \\ \hat{a}_1 \\ \dots \\ \hat{a}_n \end{pmatrix} \in IR^{(m+n)\times(m\cdot n)}$$

$$\text{Obviously, it holds: } \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} \hat{a}_i = 0 \Leftrightarrow \hat{a}_n = \sum_{i=1}^{m} a_i - \sum_{i=1}^{n-1} \hat{a}_i$$

Consequences

- Thus, we obviously can skip the last row of matrix *A*
- Note that this does not have any impact on the problem solvability since there is direct dependency between the *a*- and the *b*-vector, too
- Specifically, it holds:

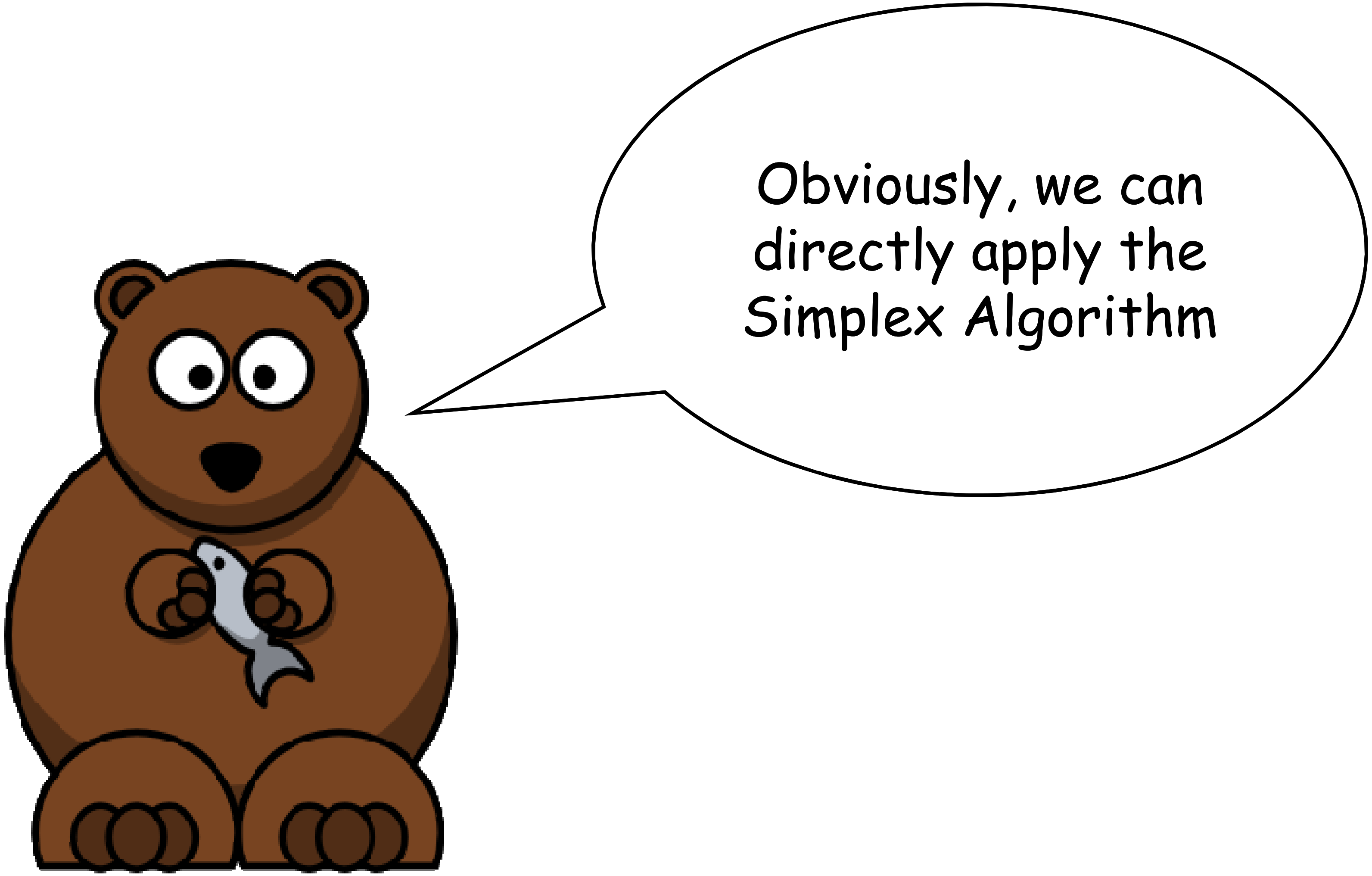
$$\boxed{\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \Leftrightarrow \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j = b_n}$$

Example

- We consider the following constellation:

$$a=\begin{pmatrix}3\\5\\6\end{pmatrix};b=\begin{pmatrix}2\\3\\6\\3\end{pmatrix};c=\begin{pmatrix}3&3&1&2\\1&2&2&3\\4&5&6&3\end{pmatrix}$$

What is to do?



Finding a feasible solution (1)

We add one slack variable per row that equals the right-hand side and has an objective function coefficient of one (comparable to the Two-Phase Method)

$$
\begin{array}{c|cccccc:cccccccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Finding a feasible solution (2)

−25	0	0	0	0	0	0	−2	−2	−2	−1	−2	−2	−2	−1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (3a)

−25	0	0	0	0	0	0	−2	−2	−2	−1	−2	−2	−2	−1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	(1)	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (3b)

−21	0	0	0	2	0	0	0	−2	−2	−1	0	−2	−2	−1	0	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (4a)

−21	0	0	0	2	0	0	0	−2	−2	−1	0	−2	−2	−1	0	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (4b)

−19	2	0	0	0	0	0	0	0	0	1	−2	−2	−2	−1	−2	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	−1	0	0	1	1	0	0	0	−1	−1	1	1	0	0	1	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (5a)

−19	2	0	0	0	0	0	0	0	0	1	−2	−2	−2	−1	−2	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	(1)	0	0	0	1	0	0	0
2	−1	0	0	1	1	0	0	0	−1	−1	1	1	0	0	1	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (5b)

-15	2	0	0	2	0	0	2	0	0	1	0	-2	-2	-1	0	-2	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	1	0	-1	0	0	-1	0	0	0	0	1	1	1	-1	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	-1	0	0	0	1	0	-1	0	-1	-1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (6a)

−15	2	0	0	2	0	0	2	0	0	1	0	−2	−2	−1	0	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	1	0	−1	0	0	−1	0	0	0	0	1	1	1	−1	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	(1)	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (6b)

−15	0	0	0	2	2	0	0	0	−2	−1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	1	1	0	−1	−1	0	0	0	1	1	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (7a)

−15	0	0	0	2	2	0	0	0	−2	−1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	1	1	0	−1	−1	0	0	0	1	1	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (7b)

−9	2	0	0	2	2	0	2	2	0	1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0

Finding a feasible solution (8a)

−9	2	0	0	2	2	0	2	2	0	1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0

Finding a feasible solution (8b)

-9	2	2	0	0	0	0	0	0	0	1	0	0	0	1	-2	-2	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	-1	-1	0	-1	-1	0	0	0	0	1	1	-1	-1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	-1	-1	0	1	1	1	0	0	0	-1	0	0	0	-1	1	1	1	0

Finding a feasible solution (9a)

−9	2	2	0	0	0	0	0	0	0	1	0	0	0	1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	−1	0	1	1	1	0	0	0	−1	0	0	0	−1	1	1	1	0

Finding a feasible solution (9b)

−5	2	2	0	2	0	0	2	0	0	1	2	0	0	1	0	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
2	0	1	0	0	−1	0	0	−1	0	0	1	0	1	1	0	−1	0	0
4	0	0	1	−1	0	0	−1	0	0	0	−1	0	0	0	0	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
1	−1	−1	0	0	1	1	−1	0	0	−1	−1	0	0	−1	0	1	1	0

Finding a feasible solution (10a)

-5	2	2	0	2	0	0	2	0	0	1	2	0	0	1	0	-2	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
2	0	1	0	0	-1	0	0	-1	0	0	1	0	1	1	0	-1	0	0
4	0	0	1	-1	0	0	-1	0	0	0	-1	0	0	0	0	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
1	-1	-1	0	0	1	1	-1	0	0	-1	-1	0	0	-1	0	(1)	1	0

Finding a feasible solution (10b)

−3	0	0	0	2	2	2	0	0	0	−1	0	0	0	−1	0	0	0	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0
3	1	1	1	−1	−1	−1	0	0	0	1	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
1	−1	−1	0	0	1	1	−1	0	0	−1	−1	0	0	−1	0	1	1	0

Finding a feasible solution (11a)

-3	0	0	0	2	2	2	0	0	0	-1	0	0	0	-1	0	0	0	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	-1	-1	0	-1	0	0	1	0	0	0	1	0
3	1	1	1	-1	-1	-1	0	0	0	1	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	-1	1	1	0	(1)	1	1	0	1	0	0	-1	0
1	-1	-1	0	0	1	1	-1	0	0	-1	-1	0	0	-1	0	1	1	0

Finding a feasible solution (11b)

−1	1	1	0	2	2	1	1	1	0	0	1	1	0	0	0	0	−1	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	1	−1	−1	0	−1	−1	0	0	−1	−1	0	0	0	0	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (12a)

−1	1	1	0	2	2	1	1	1	0	0	1	1	0	0	0	0	−1	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	(1)	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	1	−1	−1	0	−1	−1	0	0	−1	−1	0	0	0	0	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (12b)

−0	1	0	0	2	2	2	1	1	1	0	0	0	0	−1	0	0	0	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (13a)

−0	1	0	0	2	2	2	1	1	1	0	0	0	0	−1	0	0	0	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	(1)	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (13b)

−0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	−1	−1	0	−1	−1	0	0	−1	−1	0	0	0	0	1	1
5	1	0	−1	1	1	1	1	1	1	0	1	1	1	0	0	0	0	−1
0	0	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (14)

$$
\begin{array}{r|rrrrrr:rrrrrrrrrrrr}
-0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

$$\Rightarrow x_{1,1}=0, x_{1,2}=0, x_{1,3}=0, x_{1,4}=3, x_{2,1}=0, x_{2,2}=0, x_{2,3}=5, x_{2,4}=0,$$
$$x_{3,1}=2, x_{3,2}=3, x_{3,3}=1, x_{3,4}=0$$

Feasible solution

$$x_{1,1} = 0, x_{1,2} = 0, x_{1,3} = 0, x_{1,4} = 3,$$

$$x_{2,1} = 0, x_{2,2} = 0, x_{2,3} = 5, x_{2,4} = 0,$$

$$x_{3,1} = 2, x_{3,2} = 3, x_{3,3} = 1, x_{3,4} = 0,$$

$$\text{i.e., } x = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 0 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

What is to do?

Optimizing – Phase II

0	3	3	1	2	1	2	2	3	4	5	6	3
1	−1	−1	0	0	−1	−1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	−1
0	−1	−1	−1	0	0	0	0	[1]	0	0	0	1
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-6 & 1 & 1 & -1 & 0 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 3 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

−16	−1	−1	−3	0	−1	0	0	3	4	5	6	5
1	−1	−1	0	0	−1	−1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	−1
0	−1	−1	−1	0	0	0	0	[1]	0	0	0	1
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-16 & 2 & 2 & 0 & 0 & -1 & 0 & 0 & 0 & 4 & 5 & 6 & 2 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-24 & -2 & 2 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 5 & 6 & 2 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & -2 & -3 & 0 & 0 & -5 & -5 & 0 & 0 & 0 & 0 & 6 & 2 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{c|cccc|cccc|cccc}
-45 & 4 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -4 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 1a

–45	4	3	0	0	1	1	0	0	0	0	0	–4
1	–1	–1	0	0	–1	–1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	–1
0	–1	–1	–1	0	0	0	0	[1]	0	0	0	(1)
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Step 1b

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-45 & 0 & -1 & -4 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & (1) \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 1c

−45	0	−1	−4	0	1	1	0	4	0	0	0	0
1	0	0	1	0	−1	−1	0	−1	0	0	[1]	0
5	0	0	0	0	1	1	[1]	1	0	0	0	0
0	−1	−1	−1	0	0	0	0	1	0	0	0	[1]
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Step 2a

$$
\begin{array}{c|cccc|cccc|cccc}
-45 & 0 & -1 & -4 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & (1) & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 2b

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & (1) & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 2c

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & [1] & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 3a

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & (1) & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & [1] & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 3b

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & (1) & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 3c

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & [1] & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4a

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & [1] & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & (1) & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4b

$$
\begin{array}{c|cccc|cccc|cccc}
-35 & 3 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 2 & 0 & 3 & 0 \\
\hline
3 & 1 & 0 & [1] & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 & [1] & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & (1) & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4c

−35	3	0	0	1	0	−2	0	1	2	0	3	0
3	1	0	[1]	0	0	−1	0	−1	1	0	1	0
3	−1	0	0	0	0	1	[1]	1	−1	0	0	0
3	0	0	0	1	0	0	0	1	0	0	0	[1]
2	1	0	0	0	[1]	0	0	0	1	0	0	0
0	0	[1]	0	1	0	1	0	1	−1	0	−1	0
3	0	0	0	−1	0	0	0	−1	1	[1]	1	0

Optimizing – Phase II – Step 5a

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-35 & 3 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 2 & 0 & 3 & 0 \\
\hline
3 & 1 & 0 & [1] & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 & [1] & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & (1) & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 5b

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-35 & 3 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \\
\hline
3 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & (1) & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 5c

$$
\begin{array}{r|cccc:cccc:cccc}
-35 & 3 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \\
\hline
3 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & [1] & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimal solution

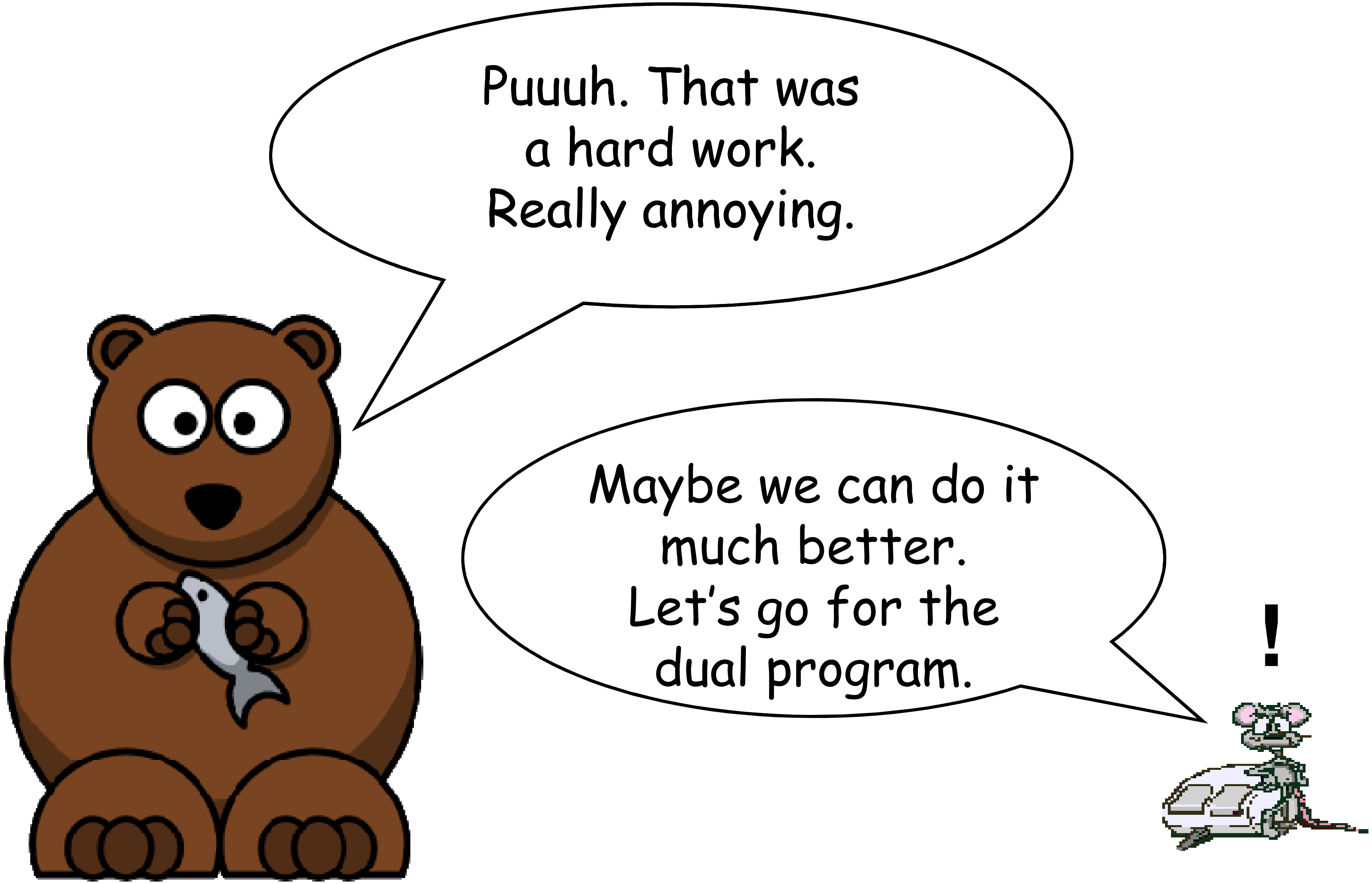
$$x_{1,1} = 0, x_{1,2} = 0, x_{1,3} = 3, x_{1,4} = 0,$$

$$x_{2,1} = 2, x_{2,2} = 0, x_{2,3} = 3, x_{2,4} = 0,$$

$$x_{3,1} = 0, x_{3,2} = 3, x_{3,3} = 0, x_{3,4} = 3,$$

$$\text{i.e., } x = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix}$$

And now?


Puuuh. That was a hard work. Really annoying.
Maybe we can do it much better. Let's go for the dual program.
!

4.2 The MODI Algorithm

- As we have seen already, the reduced costs are easy to compute for the Transportation Problem
- Thus, in what follows, we analyze them more in detail
- Here, a direct connection to the dual program is used

Calculating reduced costs

It holds:

$$\bar{c}^T = c^T - c_B^T \cdot A_B^{-1} \cdot A = c^T - \pi^T \cdot A \Rightarrow \bar{c}_{i,j} = c_{i,j} - \alpha_i - \beta_j$$

Let us assume: $\forall (i,j) \in B : \bar{c}_{i,j} = c_{i,j} - \alpha_i - \beta_j = 0 \wedge x$ bfs

$$\Rightarrow Z(x) = \sum_{(i,j)\in B} c_{i,j} \cdot x_{i,j} = \sum_{(i,j)\in B} (\alpha_i + \beta_j) \cdot x_{i,j} = \sum_{(i,j)\in B} (\alpha_i \cdot x_{i,j} + \beta_j \cdot x_{i,j}) =$$

$$\sum_i \sum_{j:(i,j)\in B} \alpha_i \cdot x_{i,j} + \sum_j \sum_{i:(i,j)\in B} \beta_j \cdot x_{i,j} = \sum_i \alpha_i \cdot \left(\sum_{j:(i,j)\in B} x_{i,j} \right) + \sum_j \beta_j \cdot \left(\sum_{i:(i,j)\in B} x_{i,j} \right) =$$

$$\sum_i \alpha_i \cdot (a_i) + \sum_j \beta_j \cdot (b_j) = b^T \cdot \pi$$

if π is feasible $\Rightarrow x, \pi$ are optimal!

(Later we will see that this procedure is a direct application of the Theorem of Complementary Slackness.)

Basic structure of the algorithm

- Start with a primal solution that is based on a basis B
- Generate a corresponding dual solution. This solution is characterized by the fact that whenever (i,j) belongs to basis B, the respective entries α_i and β_j are defined so that $\alpha_i+\beta_j=c_{i,j}$ holds
- As long as (i,j) exists with $\alpha_i+\beta_j>c_{i,j}$, find a cyclical exchange flow that reduces either α_i or β_j

Generating an initial solution

- In the following, we make use of the well-known *Northwest Corner Method*
- It generates a basic feasible solution by conducting the following steps
  1. Start with the northwest corner cell of the matrix.
  2. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.
  3. Cross out the row or column with zero supply or demand in order to indicate that no further assignment can be made in that row or column. If both, a row and a column, are set to zero simultaneously, cross out only one, and leave a zero supply or demand in the uncrossed row or column.
  4. If exactly one row or column is left out uncrossed, stop. Otherwise, move to the cell to the right if a column has just crossed, or move below if it was a row. Proceed with step 2.

The MODI Algorithm – Example

- We consider the following constellation

$$a = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}; b = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 3 \end{pmatrix}; c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 3 \end{pmatrix}$$

Example – Initial solution

$$c=\left|\begin{array}{ccccc} 2 & 3 & 6 & 3 & \\ (0) & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 3 & 6 & 3 & \\ 2 & (0) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 2 & 6 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & (0) & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 6 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & (0) & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 3 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & (0) & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 0 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & (0) & 3 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right| \Rightarrow x=\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

The dual variables

$$x=\begin{pmatrix}2&1&0&0\\0&2&3&0\\0&0&3&3\end{pmatrix}, c=\begin{pmatrix}3&3&&\\&2&2&\\&&6&3\end{pmatrix}$$

We commence with $\alpha_1=0$. Thus, we get

$$\left|\begin{matrix}\beta_1=? & \beta_2=? & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{matrix}\right| \Rightarrow \left|\begin{matrix}\beta_1=3 & \beta_2=? & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{matrix}\right| \Rightarrow$$

$$\left|\begin{matrix}\beta_1=3 & \beta_2=3 & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{matrix}\right| \Rightarrow \left|\begin{matrix}\beta_1=3 & \beta_2=3 & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=-1\\ & & 6 & 3 & \alpha_3=?\end{matrix}\right|$$

The dual variables

$$\left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=? \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=3 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=0 & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=3 \end{array}\right|$$

The resulting reduced costs

- Eventually, we obtain the tableau

$$\left| \begin{array}{ccccc} \beta_1 = 3 & \beta_2 = 3 & \beta_3 = 3 & \beta_4 = 0 & \\ 0 & 0 & -2 & 2 & \alpha_1 = 0 \\ -1 & 0 & 0 & 4 & \alpha_2 = -1 \\ -2 & -1 & 0 & 0 & \alpha_3 = 3 \end{array} \right|$$

MODI execution

- While negative reduced cost entries exist,
  - Determine the smallest reduced cost entry (i,j)
  - Insert the corresponding x-variable into the basis
    - I.e., we have to find a closed loop between the current basis members
    - Then the maximum amount is transferred along this cyclical path
    - Consequently, one element leaves the basis while (i,j) enters it
    - Correct the dual variables accordingly
- Optimal solution found

MODI Algorithm I

1. Find a feasible initial solution to the TPP
2. Determine a dual solution:
   - Set an arbitrary dual variable to ZERO
   - Calculate $\alpha_i = c_{ij} - \beta_j$ for a given β_j or $\beta_j = c_{ij} - \alpha_i$ for a given α_i
   - Use only those cost coefficients for the calculation, where the corresponding primal variable is a basis variable at that time.
3. Calculate the reduced costs $\overline{c}_{ij}$ for all non-basic variables by $\overline{c}_{ij} = c_{ij} - \alpha_i - \beta_j$
4. If $\overline{c}_{ij} \geq 0 \; \forall i, j$, then terminate since the optimal solution is found
5. Otherwise, conduct a basis change (see next slide)

MODI Algorithm II

5. Conduct a basis change
   - Choose the smallest reduced costs $\bar{c}_{pq} = \min\left\{\bar{c}_{ij} \middle| \bar{c}_{ij} < 0 \wedge \forall i, j\right\}$
   - Find a closed loop of basic variables that includes x_{pq}
   - Label x_{pq} with $+\Delta$ and label the remaining basic variables in the circle alternately with $-\Delta$ and $+\Delta$
   - Determine an upper bound
     $x_{ab} = \min\left\{x_{ij} \middle| (i, j) \text{ is a member of the closed loop and is labeled with } -\Delta\right\}$
   - x_{pq} enters the basis and x_{ab} becomes a non-basic variable
   - Calculate new values for all basic variables in the closed loop according to the labels $x_{ij} := x_{ij} \pm \Delta$
   - Calculate the objective function value $Z := Z + \bar{c}_{pq} \cdot x_{pq}$
   - Go to step 2.

The Tableau – Step 1

- Consider the initial solution

$$
\begin{array}{l|cccccc|l}
\beta = & 3 & 3 & 3 & 0 & & & \\
b = & 2 & 3 & 6 & 3 & & & \\
 & (2)^0 & (1)^0 & 0^{-2} & 0^2 & 3 & 0 & \\
 & 0^{-1} & (2)^0 & (3)^0 & 0^4 & 5 & -1 & \Rightarrow Z = 6+3+4+6+18+9=46 \\
 & 0^{-2} & 0^{-1} & (3)^0 & (3)^0 & 6 & 3 & \\
 & & & & & a & \alpha &
\end{array}
$$

The Tableau – Step 2

$$
\left|\begin{array}{lcccccc}
\beta = & 3 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 2^{0} & 1^{0} & 0^{-2} & 0^{2} & 3 & 0 \\
 & 0^{-1} & 2^{0} & 3^{0} & 0^{4} & 5 & -1 \\
 & \left[0^{-2}\right] & 0^{-1} & 3^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 3 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & \left(2^{0}\right) & \left[1^{0}\right] & 0^{-2} & 0^{2} & 3 & 0 \\
 & 0^{-1} & \left(2^{0}\right) & \left[3^{0}\right] & 0^{4} & 5 & -1 \\
 & \left[0^{-2}\right] & 0^{-1} & \left(3^{0}\right) & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$$
\left|\begin{array}{lcccccc}
\beta = & (1) & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & 3^{0} & 0^{-2} & 0^{2} & 3 & 0 \\
 & 0^{1} & 0^{0} & 5^{0} & 0^{4} & 5 & -1 \\
 & \left[2^{0}\right] & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & (3)^{0} & 0^{-2} & 0^{2} & 3 & 0 \\
 & 0^{1} & (0)^{0} & (5)^{0} & 0^{4} & 5 & -1 \\
 & (2)^{0} & 0^{-1} & (1)^{0} & (3)^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$\Rightarrow Z = 8+9+0+10+6+9 = 42$, i.e., reduction by $2 \cdot 2$

The Tableau – Step 3

$$
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & 3^{0} & \left[0^{-2}\right] & 0^{2} & 3 & 0 \\
 & 0^{1} & 0^{0} & 5^{0} & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & \left(3^{0}\right) & \left[0^{-2}\right] & 0^{2} & 3 & 0 \\
 & 0^{1} & \left[0^{0}\right] & \left(5^{0}\right) & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$$
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & 0^{0} & \left[3^{-2}\right] & 0^{2} & 3 & (0) \\
 & 0^{1} & 3^{0} & 2^{0} & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{4} & 0^{2} & \left(3^{0}\right) & 0^{4} & 3 & -2 \\
 & 0^{1} & \left(3^{0}\right) & \left(2^{0}\right) & 0^{4} & 5 & -1 \\
 & \left(2^{0}\right) & 0^{-1} & \left(1^{0}\right) & \left(3^{0}\right) & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$$\Rightarrow Z = 8+6+3+4+6+9 = 36, \text{i.e., reduction by } 3 \cdot 2$$

The Tableau – Step 4

$$
\left|\begin{array}{cccccc}
\beta = 1 & 3 & 3 & 0 & & \\
b = 2 & 3 & 6 & 3 & & \\
0^4 & 0^2 & (3^0) & 0^4 & 3 & -2 \\
0^1 & (3^0) & (2^0) & 0^4 & 5 & -1 \\
(2^0) & [0^{-1}] & (1^0) & (3^0) & 6 & 3 \\
 & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{cccccc}
\beta = 1 & 3 & 3 & 0 & & \\
b = 2 & 3 & 6 & 3 & & \\
0^4 & 0^2 & 3^0 & 0^4 & 3 & -2 \\
0^1 & (3^0) & [2^0] & 0^4 & 5 & -1 \\
2^0 & [0^{-1}] & (1^0) & 3^0 & 6 & 3 \\
 & & & & a & \alpha
\end{array}\right|
$$

$$
\Rightarrow
\left|\begin{array}{cccccc}
\beta = (1) & 3 & 3 & (0) & & \\
b = 2 & 3 & 6 & 3 & & \\
0^4 & 0^2 & 3^0 & 0^4 & 3 & -2 \\
0^1 & (2^0) & [3^0] & 0^4 & 5 & -1 \\
2^0 & [1^{-1}] & (0^0) & 3^0 & 6 & (3) \\
 & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{cccccc}
\beta = 2 & 3 & 3 & 1 & & \\
b = 2 & 3 & 6 & 3 & & \\
0^3 & 0^2 & (3^0) & 0^3 & 3 & -2 \\
0^0 & (2^0) & (3^0) & 0^3 & 5 & -1 \\
(2^0) & (1^0) & 0^1 & (3^0) & 6 & 2 \\
 & & & & a & \alpha
\end{array}\right|
$$

$\Rightarrow Z = 8+4+5+3+6+9 = 35$, i.e., reduction by $1 \cdot 1$

Optimal solution since reduced costs are non - negative!