5 The Primal-Dual Simplex Algorithm

- Again, we consider the primal program given as a minimization problem defined in standard form
- This algorithm is based on the cognition that both optimal solutions, i.e., the primal and the dual one, are strongly interdependent
- Specifically, the approach commences the searching process with a feasible dual solution and simultaneously observes the complementary slackness between the solution value of the dual and a primal solution
- If this slackness becomes zero, the optimality of the generated solutions is proven and the calculation process is terminated

Invariants of the Primal Simplex

While conducting the Primal Simplex, the following attributes are always fulfilled for a minimization problem:

(P) Minimize $c^T \cdot x$, s.t. $A \cdot x = b \wedge x \geq 0$

1. $c^T \cdot x^0 = c_B^T \cdot x_B^0 + c_N^T \cdot x_N^0 = c_B^T \cdot A_B^{-1} \cdot b = \pi^T \cdot b = b^T \cdot \pi$
2. $\overline{c}^T \cdot x^0 = \overline{c}_B^T \cdot x_B^0 + \overline{c}_N^T \cdot x_N^0 = 0 \cdot x_B^0 + \overline{c}_N^T \cdot 0 = 0,$

with $\overline{c}^T = c^T - c_B^T \cdot A_B^{-1} \cdot A$

Thus, if $\overline{c}^T \geq 0 \Rightarrow c_B^T \cdot A_B^{-1} \cdot A = \pi^T \cdot A \leq c^T \Rightarrow \pi$ is feasible for

(D) Maximize $b^T \cdot \pi$, s.t. $A^T \cdot \pi \leq c \wedge \pi$ free

Consequences

- The Primal Simplex works on a feasible primal solution that is iteratively improved by basis changes
- This is done by the consideration of a corresponding dual solution that has an identical objective function value
- As long as this dual solution is infeasible, the corresponding entries are inserted in the primal solution in order to fulfill them exactly in the dual program (→Elimination of the corresponding slackness)
- If the dual solution becomes feasible as well the optimality of both solutions (the primal and the dual solution) is proven

The Primal-Dual Simplex

- As mentioned above, we assume that the primal program is given as a minimization problem in standard form
- In what follows, we introduce a new algorithm that commences the searching process with a feasible dual solution
- This solution is analyzed according to a specific relationship to the primal problem in order to generate a corresponding primal solution that allows to prove optimality
- Specifically, we formulate a reduced problem that either generates an optimal primal solution or, if this is not possible, allows a correction of the dual one
- Obviously, this process is executed until the first case applies

Observation

5.1 Theorem of Complementary Slackness

Assuming there is a Linear Program in standard form and x and π are feasible solutions to (P) and (D), respectively.

Then, it holds:

x and π are optimal $\Leftrightarrow \left(c^T - \pi^T \cdot A\right) \cdot x = 0$

Proof of Theorem 5.1

- Fortunately, this proof is quite easy to conduct
- Based on the facts we already know about tuples of optimal primal and dual solutions, we derive

Specifically, it holds:

$$\left(c^T - \pi^T \cdot A\right) \cdot x = 0 \Leftrightarrow c^T \cdot x - \pi^T \cdot A \cdot x = 0$$

Since x is feasible for (P)

$$\Leftrightarrow c^T \cdot x - \pi^T \cdot b = 0 \Leftrightarrow c^T \cdot x = \pi^T \cdot b$$

$\Leftrightarrow$ x and π are optimal solutions

Direct consequence of Theorem 5.1

5.2 Observation

Assuming x and π are feasible solutions to (P) and (D), respectively.

Additionally, assume that it holds: $\left(c^T - \pi^T \cdot A\right) \cdot x = 0.$

Thus, x and π are optimal solutions to (P) and (D), respectively and it holds: $c^T - \pi^T \cdot A \geq 0 \wedge x \geq 0$

Hence, we can conclude

$$\left(c_j - \pi^T \cdot a^j\right) \cdot x_j = 0, \forall j \in \{1, \ldots, n\}$$

$$\Leftrightarrow x_j = 0 \vee \pi^T \cdot a^j = c_j, \forall j \in \{1, \ldots, n\}$$

A simple example

Consider the following Linear Program

$$(P)$$

$$\text{Minimize } (2,3,1,0)\cdot x$$

s.t.

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 3 & -1 \end{pmatrix} \cdot x = \begin{pmatrix} 5 \\ 9 \end{pmatrix} \wedge x \geq 0$$

Example – Thus, we get the following (D)

$$(D)$$

$$\text{Maximize } (5,9)\cdot\pi$$

$$\text{s.t.} \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \\ 0 & -1 \end{pmatrix} \cdot \pi \leq \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \wedge \ \pi \text{ free}$$

Example – How to generate π ?

Obviously, $x = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 6 \end{pmatrix}$ is a feasible solution to (P).

Thus, we have $x_1 = x_2 = 0 \wedge x_3, x_4 \neq 0$. Consequently, we need a π with the following attributes

1. $\forall i \in \{3,4\} : \pi^T \cdot a^i = c_i \Leftrightarrow \pi^T \cdot \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2. π is feasible for (D)

Example – How to generate π ?

Obviously, $\pi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ fulfills both restrictions and, therefore,

we have shown that

$$x = \begin{pmatrix} 0 \\ 0 \\ 5 \\ 6 \end{pmatrix} \text{ and } \pi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ are optimal solutions to } (P) \text{ and } (D),$$

respectively.

A feasible solution to (D)

- For what follows, we need at first a feasible solution to the dual problem. Fortunately, this is quite simple to provide.
- If c is positive, we just make use of $\pi=0$.
- Otherwise, we apply the following simple procedure that is depicted next.

Generating a feasible dual solution

- In order to generate a feasible dual solution **to cases where c≥0 does not apply**, we provide the following simple construction procedure

1. We introduce a $n+1$th variable x_{n+1} as well as a $m+1$th equality in *(P)*

$$x_1 + x_2 + \ldots + x_n + x_{n+1} = \sum_{i=1}^{n+1} x_i = b_{m+1},$$ with b_{m+1} as a huge number.

Since we add $c_{n+1} = 0$, we know that this restriction has no impact on the optimal solutions.

Generating a feasible dual solution

2. Consider now the dual program.

Maximize $b^T \cdot \pi + b_{m+1} \cdot \pi_{m+1}$

$A^T \cdot \pi + \begin{pmatrix} \pi_{m+1} & \dots & \pi_{m+1} \end{pmatrix}^T \leq c \wedge \pi_{m+1} \leq 0$, i.e.,

$\forall j : \pi^T \cdot a^j + \pi_{m+1} \leq c_j$

3. We generate $\pi^{ini} = \begin{pmatrix} \pi_1^{ini}, & \dots & ,\pi_{m+1}^{ini} \end{pmatrix}^T$ as follows:

$\pi_1^{ini} = \pi_2^{ini} = \dots = \pi_m^{ini} = 0 \wedge \pi_{m+1} = \min\left\{ c_j \mid c_j < 0 \right\} < 0$

Thus, since $j \in \{1, \dots, n\}$ exists with $c_j < 0$,

π^{ini} is feasible for (D).

The set J

Assume π to be a feasible solution to the dual program of a Linear Program in standard form.
An index $j \in \{1,\dots,n\}$ is denoted as feasible if and only if it holds: $\pi^T \cdot a^j = c_j$.

We introduce J as the set of feasible indices, i.e.,

$$J = \left\{ j \mid j \in \{1,\dots,n\} \wedge \pi^T \cdot a^j = c_j \right\}.$$

Reduced primal problem (RP)

We assume $J = \{ j_1, \ldots, j_k \}, k \geq 0$ and define

$A_J = \left(a^{j_1}, \ldots, a^{j_k} \right)$ and $x^T = \left(x^a, x^J \right)$,

with $x^a = \left(x_1^a, \ldots, x_m^a \right)$ as slack variables.

Then $\left(RP \right)$ is defined as follows:

Minimize $\xi_0 = 1^T \cdot x^a$, s.t. $\left(E_m, A_J \right) \cdot \begin{pmatrix} x^a \\ x^J \end{pmatrix} = b$, with $\begin{pmatrix} x^a \\ x^J \end{pmatrix} \geq 0$

is denoted as the reduced primal problem.

Observations

- (RP) is solvable. Specifically, we can use $x^T=(b,0)$
- Since this trivial solution has the objective function value $1^T \cdot b$ and this objective function is lower bounded by 0, (RP) is bounded
- Thus, (RP) has always a well-defined optimal solution
- Obviously, this optimal solution comprises two parts
  - First, there are the slackness variables. If these are zero, the objective function value is zero as well. Then, the primal solution is optimal to (P)
  - Secondly, there are the original variables that correspond to the set J. Since the corresponding dual values are equal to the c-vector, only these variables may become unequal to zero

Main conclusion

5.3 Theorem

(RP) always has an optimal solution. If $1^T \cdot x^a = \xi_0 = 0$,

$\begin{pmatrix} 0 \\ x^J \end{pmatrix}$ is an optimal solution to (P).

Otherwise, if $\xi_0 > 0$, then the optimal solution to the dual of (RP) always generates an improved solution to (D).

Proof of Theorem 5.3 – Case 1

At first, we assume $\xi_0 = 0$. Thus, we know that $x^a = 0$. Consequently, it holds: $A^J \cdot x^J = b$. Thus, we consider

$$\hat{x} = \begin{pmatrix} 0 \\ x^J \end{pmatrix} \geq 0 \Rightarrow A \cdot \hat{x} = b \wedge \left(c^T - \pi^T \cdot A\right) \cdot x$$

$$= \left(c^T - \pi^T \cdot A\right)_J \cdot x_J + \left(c^T - \pi^T \cdot A\right)_{J^c} \cdot x_{J^c} = 0 \cdot x_J + \left(c^T - \pi^T \cdot A\right)_{J^c} \cdot 0 = 0$$

Hence, x and π are optimal solutions to the Linear Programs (P) and (D), respectively.

Proof of Theorem 5.3 – Case 2

Now, we consider the case $\xi_0 > 0$. Thus, we know that $x^a \neq 0$. Consequently, it holds: $A^J \cdot x^J \neq b$.

Let us now consider the dual of (RP), denoted as (DRP)

(RP)

Minimize $1^T \cdot x^a$, s.t. $(E_m, A^J) \cdot \begin{pmatrix} x^a \\ x^J \end{pmatrix} = b, x^T = (x^a, x^J) \geq 0$

Thus, we obtain (DRP) as follows

Maximize $b^T \cdot \pi$, s.t. $\begin{pmatrix} E_m \\ (A^J)^T \end{pmatrix} \cdot \pi \leq \begin{pmatrix} 1^m \\ 0^{|J|} \end{pmatrix}, \pi$ free

Proof of Theorem 5.3 – Case 2

Assuming $\tilde{\pi}$ is an optimal solution to (DRP). Then, we conclude $b^T \cdot \tilde{\pi} = \xi_0 > 0$. Furthermore, let $\pi' = \pi + \lambda \cdot \tilde{\pi}$. We compute $b^T \cdot \pi' = b^T \cdot (\pi + \lambda \cdot \tilde{\pi}) = b^T \cdot \pi + b^T \cdot \lambda \cdot \tilde{\pi} =$ $b^T \cdot \pi + \lambda \cdot b^T \cdot \tilde{\pi} = b^T \cdot \pi + \lambda \cdot \xi_0 > b^T \cdot \pi$.

Consequently, if π' is feasible for (D), π' outperforms π. Hence, we now have to determine suitable values for λ resulting in feasible values for π'.

Proof of Theorem 5.3 – Case 2

Note that it holds: π' feasible for (D)

$$\Leftrightarrow \forall j : c_j - \pi'^T \cdot a^j \geq 0$$

$$\Leftrightarrow \forall j : c_j - \left(\pi + \lambda \cdot \tilde{\pi}\right)^T \cdot a^j \geq 0$$

$$\Leftrightarrow \forall j : c_j - \pi^T \cdot a^j - \lambda \cdot \tilde{\pi}^T \cdot a^j \geq 0$$

$$\Leftrightarrow \forall j : c_j - \pi^T \cdot a^j \geq \lambda \cdot \tilde{\pi}^T \cdot a^j$$

Proof of Theorem 5.3 – Case 2

$$\forall j: c_j - \pi^T \cdot a^j \geq \lambda \cdot \tilde{\pi}^T \cdot a^j$$

Since π is feasible for (D), we know that $c_j - \pi^T \cdot a^j \geq 0.$

Let us now consider the corresponding final tableau of (RP)

$\Rightarrow$

$$\begin{array}{c|ccc} 0 & 1^T & 0^T & 0^T \\ \hline b & E_m & A^J & A^{J^c} \end{array} \Rightarrow \begin{array}{c|ccc} -\xi_0 & 1^T - \tilde{\pi}^T & 0^T - \tilde{\pi}^T \cdot A^J & 0^T - \tilde{\pi}^T \cdot A^{J^c} \\ \hline \ldots & \ldots & \ldots & \ldots \end{array}$$

$$\Rightarrow 0^T - \tilde{\pi}^T \cdot A^J \geq 0 \Leftrightarrow \tilde{\pi}^T \cdot A^J \leq 0 \Rightarrow \forall j \in J: \tilde{\pi}^T \cdot a^j \leq 0$$

Hence, if $j \in J \wedge \lambda > 0,$ the feasibility restriction is always fulfilled.

Proof of Theorem 5.3 – Case 2

$$\begin{array}{c|ccc} 0 & 1^T & 0^T & 0^T \\ \hline b & E_m & A^J & A^{J^c} \end{array} \Rightarrow \begin{array}{c|ccc} -\xi_0 & 1^T - \tilde{\pi}^T & 0^T - \tilde{\pi}^T \cdot A^J & 0^T - \tilde{\pi}^T \cdot A^{J^c} \\ \hline \ldots & \ldots & \ldots & \ldots \end{array}$$

However, since A^{J^c} does not belong to (RP), there may be $j \in J^c$ with $\tilde{\pi}^T \cdot a^j > 0$. If so, we can determine

$$\bar{\lambda} = min\left\{ \frac{c_j - \pi^T \cdot a^j}{\tilde{\pi}^T \cdot a^j} \mid \forall j \in J^c : \tilde{\pi}^T \cdot a^j > 0 \right\}.$$

Consequently, $\pi + \lambda \cdot \tilde{\pi}$ is feasible for $\lambda \in \{0, \ldots, \bar{\lambda}\}$.

If there is, however, no j with $\tilde{\pi}^T \cdot a^j > 0$, (D) is unbounded and, consequently, (P) is not solvable.

Summary – The algorithm

1. We commence the searching process with a feasible solution π to the dual program (D).
2. Then, we generate the reduced Linear Program (RP(π)) and solve it optimally. Thus, we distinguish altogether three cases:

$$1.\ \xi_0 = 0$$

$$\Rightarrow \text{The tableau provides an optimal solution to } (P).$$

Summary – The algorithm

Further cases (Continuation of step 2)

2. $\xi_0 > 0 \wedge \forall j : \tilde{\pi}^T \cdot a^j \leq 0 \Rightarrow$ The primal problem (P) is not solvable.

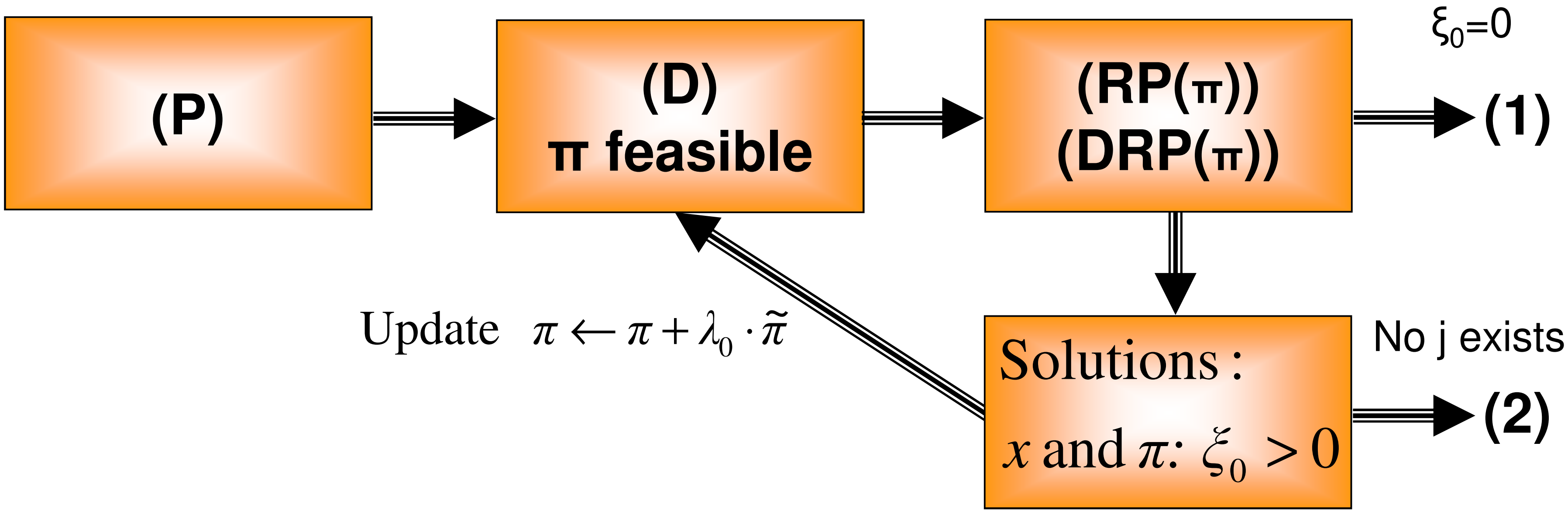
3. $\xi_0 > 0 \wedge \exists j : \tilde{\pi}^T \cdot a^j > 0$

$\Rightarrow$ Generate $\pi' = \pi + \lambda_0 \cdot \tilde{\pi}, \tilde{\pi}$ optimal solution to the problem $DRP(\pi)$.

Determine $\lambda_0 = min\left\{ \dfrac{c_j - \pi^T \cdot a^j}{\tilde{\pi}^T \cdot a^j} \mid \forall j \in J^c : \tilde{\pi}^T \cdot a^j > 0 \right\}$.

Repeat step 2 until one of the cases 1 or 2 applies.

Illustration



(1) x and π are optimal. Termination

(2) Since $b^T \cdot (\pi + \lambda_0 \cdot \tilde{\pi}) \rightarrow \infty$,

(P) is not solvable. Termination

The Primal-Dual Simplex Algorithm

1. Transform the problem such that $b \geq 0$ and generate equations.
2. Initialization with a feasible basic solution to the dual problem.
3. Determine the set $J = \left\{ j = 1,\ldots,n \mid \pi^T \cdot a^j = c_j \right\}$.
4. Solve the reduced primal problem (RP) to optimality via the Primal Simplex Algorithm:

   $$(RP)\ \xi_0 = Min\ \left(1^m\right)^T \cdot x^a \quad s.t.\ E_m \cdot x^a + A^J \cdot x^J = b \wedge x^a, x^J \geq 0$$

5. If $\xi_0 = 0$, then the optimal solution to the primal problem (P) is found. Terminate and calculate the objective function value Z with the basic variables of J: $Z = \left(c_j\right)^T_{j \in J} \cdot x^J$

Otherwise (i.e., $\xi_0 > 0$) :

6. Calculate the dual variables $\tilde{\pi}$ with cost coefficients of RP belonging to x^a: $\tilde{\pi} = 1^m - \left(\overline{c}_j\right)_{j=1,\ldots,m}$
7. If $\xi_0 > 0 \wedge \forall j : \tilde{\pi}^T \cdot a^j \leq 0$, then terminate since the primal problem (P) is unbounded and no optimal solution exists.

Otherwise (i.e., $\xi_0 > 0 \wedge \exists j : \tilde{\pi}^T \cdot a^j > 0$) :

8. Determine $\lambda_0 = min \left\{ \frac{c_j - \pi^T \cdot a^j}{\tilde{\pi}^T \cdot a^j} \middle| \forall j \notin J : \tilde{\pi}^T \cdot a^j > 0 \right\}$.
9. Update the dual variables: $\pi := \pi + \lambda_0 \cdot \tilde{\pi}$.
10. Go to step 3.

Example

(P)

$$\text{Minimize } (20,30)\cdot x, \text{s.t.}, \begin{pmatrix} 2 & 5 \\ 3 & 4 \\ 2 & 1 \end{pmatrix} \cdot x \geq \begin{pmatrix} 10 \\ 12 \\ 5 \end{pmatrix} \wedge x \geq 0$$

$\Rightarrow$

(P_s)

$$\text{Minimize } \begin{pmatrix} 20 & 30 & 0 & 0 & 0 \end{pmatrix}\cdot x, \text{s.t.}, \begin{pmatrix} 2 & 5 & -1 & 0 & 0 \\ 3 & 4 & 0 & -1 & 0 \\ 2 & 1 & 0 & 0 & -1 \end{pmatrix} \cdot x = \begin{pmatrix} 10 \\ 12 \\ 5 \end{pmatrix} \wedge x \geq 0$$

Example

(D)

$$\text{Maximize } (10,12,5,)\cdot\pi, \text{s.t.,} \begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \pi \leq \begin{pmatrix} 20 \\ 30 \\ 0 \\ 0 \\ 0 \end{pmatrix} \wedge \pi \text{ free}$$

Since $c \geq 0$, we can apply the trivial dual solution $\pi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Thus, we get $J^c = \{1,2\} \wedge J = \{3,4,5\}$

Example – Generate (RP(π))

$(RP(\pi))$

$$\text{Minimize } \begin{pmatrix}0 & 0 & 0 & 1 & 1 & 1\end{pmatrix}\cdot x, \text{ s.t. },x \geq 0 \wedge \begin{pmatrix}-1 & 0 & 0 & 1 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 0 & 0 & 1\end{pmatrix}\cdot x = \begin{pmatrix}10\\12\\5\end{pmatrix}$$

In order, however, to identify the j - values, we integrate the columns of set J^c in the tableau as well. Thus, we obtain

$$\begin{array}{c|cccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\ 12 & 0 & 1 & 0 & 3 & 4 & 0 & -1 & 0 \\ 5 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1 \end{array} \Rightarrow \begin{array}{c|cccccccc} -27 & 0 & 0 & 0 & -7 & -10 & 1 & 1 & 1 \\ \hline 10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\ 12 & 0 & 1 & 0 & 3 & 4 & 0 & -1 & 0 \\ 5 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1 \end{array}$$

Example – Generate λ_0

$$\begin{array}{r|rrrrrrrr}
-27 & 0 & 0 & 0 & -7 & -10 & 1 & 1 & 1 \\
\hline
10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\
12 & 0 & 1 & 0 & 3 & 4 & 0 & -1 & 0 \\
5 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & -1
\end{array}$$

$$\Rightarrow \begin{pmatrix}0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} - \tilde{\pi} \Leftrightarrow \tilde{\pi} = \begin{pmatrix}1 & 1 & 1\end{pmatrix}$$

$$\lambda_0 = \min\left\{\frac{20-\pi^T \cdot a^1}{\begin{pmatrix}1 & 1 & 1\end{pmatrix}\cdot a^1}, \frac{30-\pi^T \cdot a^2}{\begin{pmatrix}1 & 1 & 1\end{pmatrix}\cdot a^2}\right\} = \min\left\{\frac{20-0}{2+3+2}, \frac{30-0}{5+4+1}\right\}$$

$$= \min\left\{\frac{20}{7}, \frac{30}{10}\right\} = \frac{20}{7}$$

Example – Generate λ_0

Consequently, we obtain for the next round

$$\pi^T = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} + \frac{20}{7} \cdot \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{20}{7} & \frac{20}{7} & \frac{20}{7} \end{pmatrix}$$

Thus, since $\xi_0 = 27 > 0$, we get a new $\left(RP\left(\pi\right)\right)$

At first, we have to identify J.

Example – Generate J

Therefore, we determine

$$\pi^T = \begin{pmatrix} \frac{20}{7} & \frac{20}{7} & \frac{20}{7} \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{20}{7} \\ \frac{20}{7} \\ \frac{20}{7} \end{pmatrix} = \begin{pmatrix} 140/7 \\ 200/7 \\ -20/7 \\ -20/7 \\ -20/7 \end{pmatrix} = \begin{pmatrix} 20 \\ 200/7 \\ -20/7 \\ -20/7 \\ -20/7 \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \\ 0 \\ 0 \\ 0 \end{pmatrix} = c$$

Thus, this time we obtain $J = \{1\} \wedge J^c = \{2,3,4,5\}$

Example – Solving (RP(π))

$$\Rightarrow \begin{array}{c|cccccccc}
-27 & 0 & 0 & 0 & [-7] & -10 & 1 & 1 & 1 \\
\hline
10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\
12 & 0 & 1 & 0 & 3 & 4 & 0 & -1 & 0 \\
5 & 0 & 0 & 1 & (2) & 1 & 0 & 0 & -1
\end{array}
\Rightarrow \begin{array}{c|cccccccc}
-27 & 0 & 0 & 0 & -7 & -10 & 1 & 1 & 1 \\
\hline
10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\
12 & 0 & 1 & 0 & 3 & 4 & 0 & -1 & 0 \\
5/2 & 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 & -1/2
\end{array}$$

$$\Rightarrow \begin{array}{c|cccccccc}
-27 & 0 & 0 & 0 & -7 & -10 & 1 & 1 & 1 \\
\hline
10 & 1 & 0 & 0 & 2 & 5 & -1 & 0 & 0 \\
9/2 & 0 & 1 & -3/2 & 0 & 5/2 & 0 & -1 & 3/2 \\
5/2 & 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 & -1/2
\end{array}$$

$$\Rightarrow \begin{array}{c|cccccccc}
-19/2 & 0 & 0 & 7/2 & 0 & -13/2 & 1 & 1 & -5/2 \\
\hline
5 & 1 & 0 & -1 & 0 & 4 & -1 & 0 & 1 \\
9/2 & 0 & 1 & -3/2 & 0 & 5/2 & 0 & -1 & 3/2 \\
5/2 & 0 & 0 & 1/2 & 1 & 1/2 & 0 & 0 & -1/2
\end{array}$$

Example – Solving (RP(π))

$$\Rightarrow \left(0,0,\tfrac{7}{2}\right) = (1,1,1) - \tilde{\pi}^T \Leftrightarrow \tilde{\pi}^T = \left(1,1,-\tfrac{5}{2}\right)$$

$$\Rightarrow \lambda_0 = \min\left\{\frac{30-\frac{200}{7}}{5+4-\tfrac{5}{2}}, \frac{0-\left(-\tfrac{20}{7}\right)}{\tfrac{5}{2}}\right\} = \min\left\{\frac{\frac{10}{7}}{\tfrac{13}{2}}, \frac{\tfrac{20}{7}}{\tfrac{5}{2}}\right\}$$

$$= \min\left\{\frac{20}{91}, \frac{40}{35}\right\} = \frac{20}{91}$$

Example – Updating π and J

Consequently, we obtain for the next round

$$\pi^T = \begin{pmatrix} \frac{20}{7} & \frac{20}{7} & \frac{20}{7} \end{pmatrix} + \frac{20}{91} \cdot \begin{pmatrix} 1 & 1 & -5/2 \end{pmatrix} = \begin{pmatrix} \frac{40}{13} & \frac{40}{13} & \frac{30}{13} \end{pmatrix}$$

Thus, since $\xi_0 = 19/2 > 0$, we get a new $\left(RP(\pi)\right)$

At first, we again have to identify J.

$$\pi^T = \begin{pmatrix} \frac{40}{13} & \frac{40}{13} & \frac{30}{13} \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{40}{13} \\ \frac{40}{13} \\ \frac{30}{13} \end{pmatrix} = \begin{pmatrix} 260/13 \\ 390/13 \\ -40/13 \\ -40/13 \\ -30/13 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \\ -40/13 \\ -40/13 \\ -30/13 \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \\ 0 \\ 0 \\ 0 \end{pmatrix} = c$$

Thus, this time we obtain $J = \{1,2\} \wedge J^c = \{3,4,5\}$

Example – Solving (RP(π))

$$
\Rightarrow
\begin{array}{c|ccccccccc}
-\frac{19}{2} & 0 & 0 & \frac{7}{2} & 0 & \left[-\frac{13}{2}\right] & 1 & 1 & -\frac{5}{2} \\
\hline
5 & 1 & 0 & -1 & 0 & (4) & -1 & 0 & 1 \\
\frac{9}{2} & 0 & 1 & -\frac{3}{2} & 0 & \frac{5}{2} & 0 & -1 & \frac{3}{2} \\
\frac{5}{2} & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & -\frac{1}{2}
\end{array}
$$

$$
\Rightarrow
\begin{array}{c|ccccccccc}
-\frac{11}{8} & \frac{13}{8} & 0 & \frac{15}{8} & 0 & 0 & -\frac{5}{8} & 1 & -\frac{7}{8} \\
\hline
\frac{5}{4} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{4} \\
\frac{11}{8} & -\frac{5}{8} & 1 & -\frac{7}{8} & 0 & 0 & \frac{5}{8} & -1 & \frac{7}{8} \\
\frac{15}{8} & -\frac{1}{8} & 0 & \frac{5}{8} & 1 & 0 & \frac{1}{8} & 0 & -\frac{5}{8}
\end{array}
$$

$$\Rightarrow \left(\frac{13}{8}, 0, \frac{15}{8}\right) = (1,1,1) - \tilde{\pi}^T \Leftrightarrow \tilde{\pi}^T = \left(-\frac{5}{8}, 1, -\frac{7}{8}\right)$$

Example – Solving (RP(π))

$$\Rightarrow \left({}^{13}\!/_8, 0, {}^{15}\!/_8\right) = (1,1,1) - \tilde{\pi}^T \Leftrightarrow \tilde{\pi}^T = \left(-{}^{5}\!/_8, 1, -{}^{7}\!/_8\right)$$

$$\Rightarrow \lambda_0 = \min\left\{\frac{{}^{40}\!/_{13}}{{}^{5}\!/_8}, \frac{{}^{30}\!/_{13}}{{}^{7}\!/_8}\right\} = \frac{240}{91}$$

Thus, we can update π by : $\pi^T = \left({}^{40}\!/_{13}, {}^{40}\!/_{13}, {}^{30}\!/_{13}\right) + \frac{240}{91} \cdot \left(-{}^{5}\!/_8, 1, -{}^{7}\!/_8\right)$

$$= \left({}^{40}\!/_{13} - {}^{1200}\!/_{728}, {}^{40}\!/_{13} + {}^{240}\!/_{91}, {}^{30}\!/_{13} - {}^{1680}\!/_{728}\right)$$

$$= \left({}^{(2240-1200)}\!/_{728}, {}^{(280+240)}\!/_{91}, {}^{(1680-1680)}\!/_{728}\right)$$

$$= \left({}^{1040}\!/_{728}, {}^{520}\!/_{91}, 0\right) = \left({}^{10}\!/_7, {}^{40}\!/_7, 0\right)$$

Example – Updating J

Again, we have to identify J.

$$\pi^T = \begin{pmatrix} \frac{10}{7} & \frac{40}{7} & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 10/7 \\ 40/7 \\ 0 \end{pmatrix} = \begin{pmatrix} 140/7 \\ 210/7 \\ -10/7 \\ -40/7 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ 30 \\ -10/7 \\ -40/7 \\ 0 \end{pmatrix} \leq \begin{pmatrix} 20 \\ 30 \\ 0 \\ 0 \\ 0 \end{pmatrix} = c$$

Thus, this time we obtain $J = \{1,2,5\} \wedge J^c = \{3,4\}$

Example – Solving (RP(π))

$$
\begin{array}{c|cccccccc}
-\frac{11}{8} & \frac{13}{8} & 0 & \frac{15}{8} & 0 & 0 & -\frac{5}{8} & 1 & \left[-\frac{7}{8}\right] \\
\hline
\frac{5}{4} & \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{4} \\
\frac{11}{8} & -\frac{5}{8} & 1 & -\frac{7}{8} & 0 & 0 & \frac{5}{8} & -1 & \left(\frac{7}{8}\right) \\
\frac{15}{8} & -\frac{1}{8} & 0 & \frac{5}{8} & 1 & 0 & \frac{1}{8} & 0 & -\frac{5}{8}
\end{array}
$$

$$
\Rightarrow
\begin{array}{c|cccccccc}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\frac{6}{7} & \frac{3}{7} & -\frac{2}{7} & 0 & 0 & 1 & -\frac{3}{7} & \frac{2}{7} & 0 \\
\frac{11}{7} & -\frac{5}{7} & \frac{8}{7} & -1 & 0 & 0 & \frac{5}{7} & -\frac{8}{7} & 1 \\
\frac{20}{7} & -\frac{4}{7} & \frac{5}{7} & 0 & 1 & 0 & \frac{4}{7} & \frac{5}{7} & 0
\end{array}
$$

$\Rightarrow \xi_0 = 0 \Rightarrow$ Optimal solutions are:

$$\pi^T = \left(\frac{10}{7}, \frac{40}{7}, 0\right) \wedge x^T = \left(\frac{20}{7} \quad \frac{6}{7} \quad 0 \quad 0 \quad \frac{11}{7}\right)$$

Additional literature to Section 5

The primal-dual algorithm for general LP's was first described in

- Dantzig, G.B.: Ford, L.R.; Fulkerson, D.R. (1956): A Primal-Dual Algorithm for Linear Programs," in Kuhn, H.W.; Tucker, A.W. (eds.): *Linear Inequalities and Related Systems.* Princeton University Press, Princeton, N.J., pp. 171-181.

It is introduced there as a generalization of the paper

- Kuhn, H.W. (1955): The Hungarian Method for the Assignment Problem. *Naval Research Logistics Quarterly,* 2, nos. 1 and 2, pp. 83-97.