8 Transportation Problem – Alpha-Beta

- Now, we introduce an additional algorithm for the Hitchcock Transportation problem, which was already introduced before
- This is the Alpha-Beta Algorithm
- It completes the list of solution approaches for solving this well-known problem
- The Alpha-Beta Algorithm is a primal-dual solution algorithm
- Owing to the simplicity of the dual problem, this procedure is capable of using significant insights into the problem structure

8.1 Problem definition and analysis

Refresh: The primal problem…

$c_{i,j}$: Delivery costs for each product unit that is transported from supplier i to customer j

a_i : Total supply of $i = 1,...,m$

b_j : Total demand of $j = 1,...,n$

$x_{i,j}$: Quantity that supplier $i = 1,...,m$ delivers to the customer $j = 1,...,n$

$$(P)\text{Minimize } c^T \cdot x$$

$$\text{s.t.} \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \dots & \dots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} \cdot x = \begin{pmatrix} a_1 \\ \dots \\ \dots \\ a_m \\ b \end{pmatrix}$$

$$x = \left(x_{1,1},...,x_{1,j},...,x_{1,n},...,x_{i,1},...,x_{i,n}...,x_{m,1},...,x_{m,n}\right)^T \geq 0$$

and the corresponding dual

$$(D)\ \text{Maximize} \sum_{i=1}^{m} a_i \cdot \pi_i + \sum_{j=1}^{n} b_j \cdot \pi_{m+j} = \sum_{i=1}^{m} a_i \cdot \alpha_i + \sum_{j=1}^{n} b_j \cdot \beta_j \ \text{ s.t.}$$

$$\begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \dots & & E_n \\ & & \dots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \pi \leq \begin{pmatrix} c_{1,1} \\ \dots \\ c_{i,1} \\ \dots \\ c_{m,n} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \dots & & E_n \\ & & \dots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \begin{pmatrix} c_{1,1} \\ \dots \\ c_{i,1} \\ \dots \\ c_{m,n} \end{pmatrix},$$

i.e.,

$$\forall i \in \{1,...,n\} : \forall j \in \{1,...,m\} : \alpha_i + \beta_j \leq c_{i,j}$$

Direct Observation

- The dual considers a somewhat modified problem
- This may be interpreted as follows
  - There is a third party that offers transportation service between the plants and the consumers
  - For this service, both sides have to pay an individual fee. Specifically, the *i*th supplier pays α_i and the *j*th consumer β_j
  - Obviously, it is not possible to charge more than $c_{i,j}$ for the respective combination
  - Otherwise, since it possesses a more efficient alternative, the company would not make use of this alternative
  - Thus, the difference $c_{i,j}$- α_i- β_j is denoted as a speculative gain of the considered company
  - Consequently, whenever this difference is negative, the primal problem is hold to introduce (i,j) in the basis. Otherwise, we better keep it out.

The first row of the primal tableau

If we consider the first row of the primal tableau, we directly obtain

$$\overline{c}_{i,j} = c_{i,j} - c_B \cdot A_B^{-1} \cdot A = c_{i,j} - \pi^T \cdot A = c_{i,j} - A^T \cdot \pi$$
$$= c_{i,j} - \alpha_i - \beta_j$$

If we have $\overline{c}_{i,j} < 0$, the dual variables are not feasible and outsourcing is not reasonable.

Feasible dual solutions

Obviously, since $c_{i,j} \geq 0$, we have $\pi = 0^{n+m}$ as a trivial initial solution.

This trivial solution can be directly improved by

$$\beta_j = \min\left\{c_{i,j} \mid i = 1,...,m\right\}$$
$$\wedge \alpha_i = \min\left\{c_{i,j} - \beta_j \mid j = 1,...,n\right\}$$

Consider an example

$$a^T = (3 \quad 5 \quad 6) \wedge b^T = (2 \quad 3 \quad 6 \quad 3) \wedge c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 3 \end{pmatrix}$$

$\Rightarrow$

Generating an initial solution :

$\beta = (1 \quad 2 \quad 1 \quad 2)^T \Rightarrow$

$$\alpha = \begin{pmatrix} \min\{3-1, 3-2, 1-1, 2-2\} \\ \min\{1-1, 2-2, 2-1, 3-2\} \\ \min\{4-1, 5-2, 6-1, 3-2\} \end{pmatrix} = \begin{pmatrix} \min\{2,1,0,0\} \\ \min\{0,0,1,1\} \\ \min\{3,3,5,1\} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Example

With $\alpha = (0 \quad 0 \quad 1)^T \wedge \beta = (1 \quad 2 \quad 1 \quad 2)^T$, we get

$$\bar{c} - (\alpha \quad \alpha \quad \alpha \quad \alpha) - \begin{pmatrix} \beta^T \\ \beta^T \\ \beta^T \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{pmatrix} \geq 0.$$ Thus, the solution is obviously feasible

Preparing the Primal-Dual Algorithm

In order to prepare the Primal-Dual Algorithm, we introduce:

$IJ = \{(i,j) \mid \alpha_i + \beta_j = c_{i,j}\}$. Thus, we obtain the reduced primal *(RP)*

Minimize $1^T \cdot x^a$, s.t.,

$$\left(E_{(n+m)}, A^{(IJ)}\right) \cdot \begin{pmatrix} x^a \\ x^{(IJ)} \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, a \in IR^m, b \in IR^n$$

$$\wedge x^a \geq 0 \wedge x^{(IJ)} \geq 0$$

$$\Leftrightarrow \text{Minimize } \sum_{i=1}^{n+m} x_i^a, \text{ s.t.,}$$

$$x_i^a + \sum_{j \mid (i,j) \in IJ} a_{i,j} \cdot x_{i,j} = a_i, \forall i \in \{1,...,m\}$$

$$\wedge x_{j+m}^a + \sum_{i \mid (i,j) \in IJ} a_{i,j} \cdot x_{i,j} = b_j, \ \forall j \in \{1,...,n\} \wedge x^a \geq 0 \wedge x^{(IJ)} \geq 0$$

Preparing the Primal-Dual Algorithm

$$\Leftrightarrow$$

$$\text{Minimize} \sum_{i=1}^{n+m} x_i^a,$$

s.t.,

$$x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j} = a_i, \forall i \in \{1,\dots,m\}$$

$$\wedge x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j} = b_j, \forall j \in \{1,\dots,n\}$$

$$\wedge x^a \geq 0 \wedge x^{(IJ)} \geq 0$$

8.2 Analyzing the reduced primal (RP)

Obviously, it holds:

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m}\left(x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j}\right) \wedge \sum_{j=1}^{n} b_j = \sum_{j=1}^{n}\left(x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j}\right)$$

Since total demand and supply are identical, we have

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \Leftrightarrow \sum_{i=1}^{m}\left(x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j}\right) = \sum_{j=1}^{n}\left(x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j}\right)$$

$$\Leftrightarrow \sum_{i=1}^{m} x_i^a + \sum_{i=1}^{m}\sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a + \sum_{j=1}^{n}\sum_{i|(i,j)\in IJ} x_{i,j}$$

Analyzing (RP)

$$\sum_{i=1}^{m} x_i^a + \sum_{i=1}^{m}\sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a + \sum_{j=1}^{n}\sum_{i|(i,j)\in IJ} x_{i,j}$$

$$\text{Obviously, it holds}: \sum_{i=1}^{m}\sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n}\sum_{i|(i,j)\in IJ} x_{i,j}$$

Hence, we conclude:

$$\sum_{i=1}^{m} x_i^a + \sum_{i=1}^{m}\sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a + \sum_{j=1}^{n}\sum_{i|(i,j)\in IJ} x_{i,j}$$

$$\Leftrightarrow \sum_{i=1}^{m} x_i^a = \sum_{j=1}^{n} x_{j+m}^a$$

Direct conclusion

Altogether, we therefore obtain:

$$\sum_{i=1}^{m} x_i^a + \sum_{i=1}^{m} \sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{i=1}^{m} x_i^a + \sum_{(i,j)\in IJ} x_{i,j} = \sum_{i=1}^{m} a_i$$

$$\Leftrightarrow \sum_{i=1}^{m} x_i^a = \sum_{i=1}^{m} a_i - \sum_{(i,j)\in IJ} x_{i,j}$$

$$\sum_{j=1}^{n} x_{j+m}^a + \sum_{j=1}^{n} \sum_{i|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a + \sum_{(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} b_j$$

$$\Leftrightarrow \sum_{j=1}^{n} x_{j+m}^a = \sum_{j=1}^{n} b_j - \sum_{(i,j)\in IJ} x_{i,j} \Rightarrow \sum_{i=1}^{m+n} x_i^a = \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j - 2\cdot \sum_{(i,j)\in IJ} x_{i,j}$$

Consequences

Since minimizing $\sum_{i=1}^{m+n} x_i^a = \sum_{i=1}^{m} a_i + \sum_{j=1}^{n} b_j - 2\cdot \sum_{(i,j)\in IJ} x_{i,j}$ determines the objective function of the reduced primal of the Hitchcock Transportation Problem, we just have to maximize $2\cdot \sum_{(i,j)\in IJ} x_{i,j}$

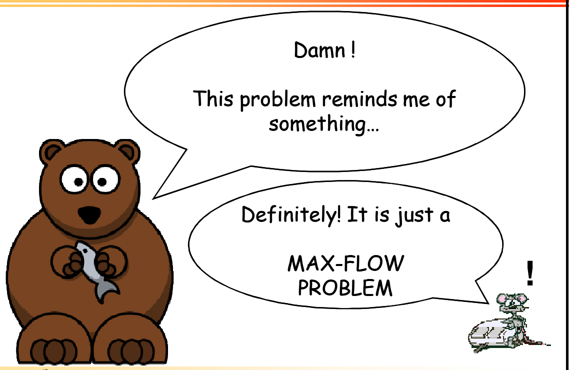
This leads to the following (RP):

Maximize $\sum_{(i,j)\in IJ} x_{i,j}$,

s.t.,

$$x_{i,j} \geq 0, \forall i,j \wedge \sum_{j|(i,j)\in IJ} x_{i,j} \leq a_i, \forall i \in \{1,...,m\} \wedge \sum_{i|(i,j)\in IJ} x_{i,j} \leq b_j, \forall j \in \{1,...,n\}$$

Analyzing the problem in detail



The RP is a specific Flow Problem

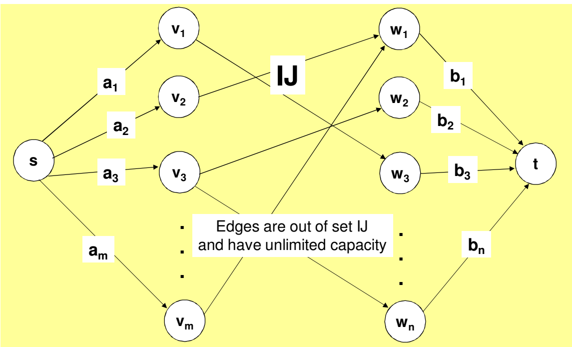
Obviously, the problem (RP) can be modeled as a Max-Flow Problem.
For this purpose, we define the following network:

$$V=\{s,v_1,\ldots,v_m,w_1,\ldots,w_n,t\}$$

$$E=\{(s,v_i)\mid 1\leq i\leq m\}\cup\{(v_i,w_j)\mid 1\leq i\leq m\wedge 1\leq j\leq n\wedge(i,j)\in IJ\}$$
$$\cup\{(w_j,t)\mid 1\leq j\leq n\}$$

$$c(s,v_i)=a_i,\forall i\in\{1,\ldots,m\}\wedge c(v_i,w_j)=\infty,\forall(i,j)\in IJ$$
$$\wedge c(w_j,t)=b_j,\forall j\in\{1,\ldots,n\}$$

Illustration of the network



Resuming with our example

- In the example introduced above, we generated the following initial solution

$$\alpha=\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T\wedge\beta=\begin{pmatrix}1 & 2 & 1 & 2\end{pmatrix}^T$$

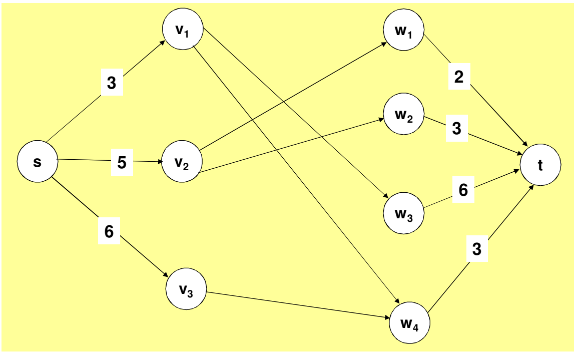
- Thus, we can derive

With $\alpha=\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T\wedge\beta=\begin{pmatrix}1 & 2 & 1 & 2\end{pmatrix}^T$

we obtain the reduced matrix: $\begin{pmatrix}2 & 1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 2 & 2 & 4 & 0\end{pmatrix}$

$$\Rightarrow IJ=\{(1,3),(1,4),(2,1),(2,2),(3,4)\}$$

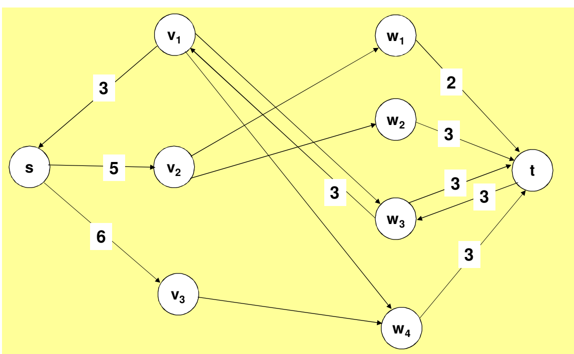
We obtain the following network



Augmenting the flow

- At first, we find the flow
  - s-v_1-w_3-t
  - It can be augmented up to 3
- Therefore, we update the network…

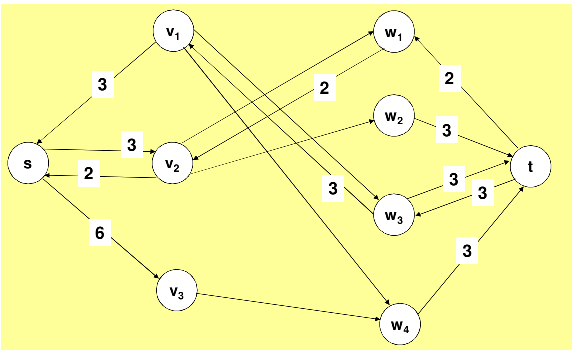
We obtain the modified network



Augmenting the flow

- Now, we find
  - s-v_2-w_1-t
  - It can be augmented up to 2
- Therefore, we update the network…

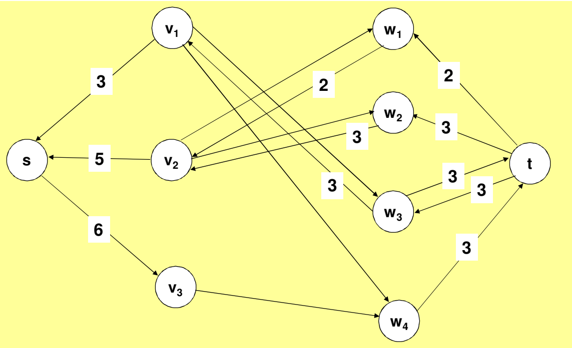
Illustration



Augmenting the flow

- Now, we find
  - s-v_2-w_2-t
  - It can be augmented up to 3
- Therefore, we update the network…

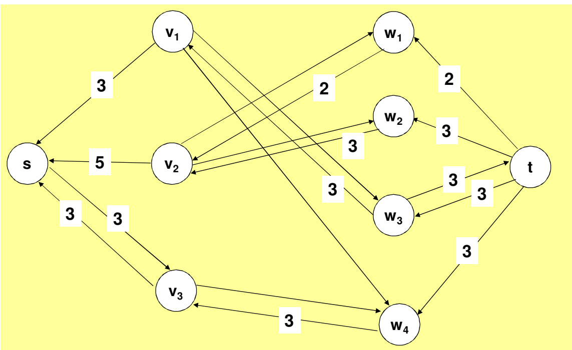
Modifying our network again



Augmenting again the flow

- Now, we find
  - s-v_3-w_4-t
  - It can be augmented up to 3
- Therefore, we update the network…

And the network is adjusted to



Solution to the reduced primal problem

Thus, we obtain :

$$x=\begin{pmatrix}0 & 0 & 3 & 0\\ 2 & 3 & 0 & 0\\ 0 & 0 & 0 & 3\end{pmatrix}. \text{ Obviously } x \text{ is not feasible for } (P)$$

Owing to the vectors $a^T=(3 \quad 5 \quad 6)\wedge$

$b^T=(2 \quad 3 \quad 6 \quad 3)$, we need the vector of slackness

variables $x^a=(0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 3 \quad 0)^T$

Updating the dual solution

- Obviously, we can optimally solve (RP) by making use of an efficient Max-Flow Algorithm
- Unfortunately, this does not provide a mechanism for updating the dual solution yet
- In order to do so, we have to analyze the dual of the reduced primal *(DRP)*

Modified Reduced Primal (RP_1)

$$\text{Minimize } \sum_{i=1}^{n+m} x_i^a,$$

s.t.,

$$x_{i,j} \geq 0, \forall i,j \wedge x_i^a \geq 0, \forall i \in \{1,\ldots,n+m\} \wedge$$

$$x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j} = a_i, \forall i \in \{1,\ldots,m\} \wedge$$

$$x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j} = b_j, \forall j \in \{1,\ldots,n\}$$

Modified Reduced Primal (RP$_1$)

Since it holds

$$\sum_{i=1}^{m}\left(x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j}\right) = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = \sum_{j=1}^{n}\left(x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j}\right)$$

$$\Leftrightarrow \sum_{i=1}^{m} x_i^a + \sum_{i=1}^{m}\sum_{j|(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a + \sum_{j=1}^{n}\sum_{i|(i,j)\in IJ} x_{i,j}$$

$$\Leftrightarrow \sum_{i=1}^{m} x_i^a + \sum_{(i,j)\in IJ} x_{i,j} = \sum_{j=1}^{n} x_{j+m}^a \sum_{(i,j)\in IJ} x_{i,j} \Leftrightarrow \sum_{i=1}^{m} x_i^a = \sum_{j=1}^{n} x_{j+m}^a \Leftrightarrow 2\cdot\sum_{i=1}^{m} x_i^a = \sum_{i=1}^{m+n} x_i^a$$

Thus, we obtain the equivalent problem:

Minimize $\sum_{i=1}^{m} x_i^a$, s.t., $x_{i,j} \geq 0, \forall i,j \wedge x_i^a \geq 0, \forall i \in \{1,...,n+m\} \wedge$

$x_i^a + \sum_{j|(i,j)\in IJ} x_{i,j} = a_i, \forall i \in \{1,...,m\} \wedge x_{j+m}^a + \sum_{i|(i,j)\in IJ} x_{i,j} = b_j, \forall j \in \{1,...,n\}$

...and its dual counterpart (DRP$_1$)

Maximize $\sum_{i=1}^{m} a_i \cdot \alpha_i + \sum_{j=1}^{n} b_j \cdot \beta_j$

s.t.,

$\alpha_i + \beta_j \leq 0, \forall (i,j) \in IJ$

$\alpha_i \leq 1, \forall i \in \{1,...,m\} \wedge \beta_j \leq 0, \forall j \in \{1,...,n\}$

8.3 Solving the DRP

8.3.1 Theorem

Assuming (RP) was optimally solved by an appropriate Max-Flow Algorithm. Furthermore, (W,W^c) is the resulting s-t-cut according to the current x with $W = \{v \in V \mid v \text{ is reachable from } s \text{ in the final network of } (RP)\}$.

Then,

$\tilde{\alpha}_i = \begin{cases} 1 & \text{if } v_i \in W \\ 0 & \text{if } v_i \in W^c \end{cases} \wedge \tilde{\beta}_j = \begin{cases} -1 & \text{if } w_j \in W \\ 0 & \text{if } w_j \in W^c \end{cases}$ determines an optimal solution for (DRP_1). Additionally, $(\hat{\alpha},\hat{\beta})$, with $\hat{\alpha} = \alpha + \lambda_0 \cdot \tilde{\alpha} \wedge$

$\hat{\beta} = \beta + \lambda_0 \cdot \tilde{\beta}$ are improved solutions of (D) if $\sum_{i=1}^{n+m} x_i^a > 0 \wedge \lambda_0 > 0$

Proof of the Theorem – Basic cognitions

As a preliminary step, we generate some basic attributes

1. If $v_i \in W$, we know that:
if additionally $(i,j) \in IJ \Rightarrow w_j \in W$
This results from the following observation:
If $v_i \in W \wedge (i,j) \in IJ$, then we know that there is an edge with unlimited capacity connecting v_i and w_j. Hence, it holds $c_{i,j} > f_{i,j}$ and therefore w_j is reachable from s as well.

Proof of the Theorem – Basic cognitions

2. Corollary:
$v_i \in W \wedge w_j \in W^c \Rightarrow (i,j) \notin IJ$

3. If $w_j \in W \wedge (i,j) \in IJ \wedge x_{i,j} > 0 \Rightarrow v_i \in W$
This results from the following observation:
Since $x_{i,j} > 0$, a former step has established a connection between v_i and w_j. Thus we have a backward link from w_j to v_i with capacity $x_{i,j} > 0$.

Proof of the Theorem – Basic cognitions

4. Corollary: $v_i \in W^c \wedge w_j \in W \Rightarrow (i,j) \notin IJ \vee x_{i,j} = 0$
In what follows, $r_{i,j}$ denotes the remaining capacity on the link (i,j), with
$(i,j) \in \left\{(v_i, w_j) \mid (i,j) \in IJ\right\} \cup \left\{(s, v_i) \mid i \in \{1,\ldots,m\}\right\} \cup \left\{(w_j, t) \mid j \in \{1,\ldots,n\}\right\}.$

5. $v_i \in W^c \Rightarrow r_{s,v_i} = 0 \Rightarrow \sum_{j \mid (i,j) \in E} x_{i,j} = a_i \Rightarrow x_i^a = 0$

6. $w_j \in W \Rightarrow r_{w_j,t} = 0 \Rightarrow \sum_{i \mid (i,j) \in E} x_{i,j} = b_j \Rightarrow x_{j+m}^a = 0$

Proof of Theorem 8.3.1 – Feasibility

We are now ready to commence the proof. At first, we show the feasibility of the generated solution to (DRP).
Obviously, it holds:

1. $\tilde{\alpha}_i \leq 1, \forall i \in \{1,...,m\} \wedge \tilde{\beta}_j \leq 0, \forall j \in \{1,...,n\}$

Additionally, we have to show

2. $\tilde{\alpha}_i + \tilde{\beta}_j \leq 0, \forall (i,j) \in IJ$.

2.1 $v_i \in W \Rightarrow w_j \in W \Rightarrow \tilde{\alpha}_i = 1 \wedge \tilde{\beta}_j = -1 \Rightarrow \tilde{\alpha}_i + \tilde{\beta}_j = 0$

2.2 $v_i \in W^c \Rightarrow \tilde{\alpha}_i = 0 \Rightarrow \tilde{\alpha}_i + \tilde{\beta}_j \leq 0$

Thus, $(\tilde{\alpha}_i, \tilde{\beta}_j)$ is a feasible solution to (DRP).

Proof of Theorem 8.3.1 – Optimality

We know that the optimal solution to the reduced primal problem is generated by the Max-Flow procedure and is therefore defined by the following variables
$x_{i,j}, \forall i,j \in IJ \wedge x_i^a, \forall i \in \{1,...,n+m\}$

Consequently, its objective function value is determined by $\sum_{i=1}^{m} x_i^a$

We calculate: $\sum_{i=1}^{m} a_i \cdot \tilde{\alpha}_i + \sum_{j=1}^{n} b_j \cdot \tilde{\beta}_j = \sum_{v_i \in W} a_i - \sum_{w_j \in W} b_j =$

$$\sum_{v_i \in W}\left(\sum_{j|(i,j)\in IJ} x_{i,j}\right) + \sum_{v_i \in W} x_i^a - \left(\sum_{w_j \in W}\left(\sum_{i|(i,j)\in IJ} x_{i,j}\right) + \sum_{w_j \in W} x_{j+m}^a\right)$$

RP and DRP have identical objective values

And thus, it holds:

$$\sum_{i=1}^{m} a_i \cdot \alpha_i + \sum_{j=1}^{n} b_j \cdot \beta_j =$$

$$\sum_{v_i \in W}\left(\sum_{j|(i,j)\in IJ} x_{i,j}\right) + \sum_{v_i \in W} x_i^a - \sum_{w_j \in W}\left(\sum_{i|(i,j)\in IJ} x_{i,j}\right) - \sum_{w_j \in W} x_{j+m}^a =$$

$$\sum_{(i,j)\in IJ} x_{i,j} - \sum_{(i,j)\in IJ} x_{i,j} + \sum_{v_i \in W} x_i^a - \sum_{w_j \in W} x_{j+m}^a = \sum_{v_i \in W} x_i^a - \underbrace{\sum_{w_j \in W} x_{j+m}^a}_{\text{Owing to attribute 6, this is equal to 0}}$$

$$= \sum_{v_i \in W} x_i^a = \sum_{i=1}^{m} x_i^a$$

Thus, $(\tilde{\alpha}, \tilde{\beta})$ is an optimal solution to (DRP_1)

Feasibility of the updated dual solution

We calculate $\left(\hat{\alpha},\hat{\beta}\right)=\left(\alpha,\beta\right)+\lambda_0\cdot\left(\tilde{\alpha},\tilde{\beta}\right)$

It has to be guaranteed

$$\alpha_i+\lambda_0\cdot\tilde{\alpha}_i+\beta_j+\lambda_0\cdot\tilde{\beta}_j\le c_{i,j}\Leftrightarrow \alpha_i+\beta_j+\lambda_0\cdot\tilde{\alpha}_i+\lambda_0\cdot\tilde{\beta}_j\le c_{i,j}$$

$$\lambda_0\cdot\left(\tilde{\alpha}_i+\tilde{\beta}_j\right)\le c_{i,j}-\alpha_i-\beta_j\Leftrightarrow \lambda_0\le\frac{c_{i,j}-\alpha_i-\beta_j}{\tilde{\alpha}_i+\tilde{\beta}_j}$$

$$\tilde{\alpha}_i+\tilde{\beta}_j=\begin{cases}0 & \text{if } v_i\in W\wedge w_j\in W\\ 0 & \text{if } v_i\in W^c\wedge w_j\in W^c\\ -1 & \text{if } v_i\in W^c\wedge w_j\in W\\ 1 & \text{if } v_i\in W\wedge w_j\in W^c\end{cases}=\begin{cases}1 & \text{if } v_i\in W\wedge w_j\in W^c\\ -1 & \text{if } v_i\in W^c\wedge w_j\in W\\ 0 & \text{otherwise}\end{cases}$$

Proof of Theorem 8.3.1 – Defining λ_0

$$\lambda_0\le\frac{c_{i,j}-\alpha_i-\beta_j}{\tilde{\alpha}_i+\tilde{\beta}_j},\text{ with }\tilde{\alpha}_i+\tilde{\beta}_j=\begin{cases}1 & \text{if } v_i\in W\wedge w_j\in W^c\\ -1 & \text{if } v_i\in W^c\wedge w_j\in W\\ 0 & \text{otherwise}\end{cases}$$

If $(i,j)\in IJ$, we have to consider the case $v_i\in W^c\wedge w_j\in W$

$$\Rightarrow \lambda_0\ge\min\left\{\alpha_i+\beta_j-c_{i,j}\mid (i,j)\in IJ\wedge v_i\in W^c\wedge w_j\in W\right\}\le 0$$

If $(i,j)\notin IJ$, we have to consider the case $v_i\in W\wedge w_j\in W^c$

$$\Rightarrow \lambda_0\le\min\left\{c_{i,j}-\alpha_i-\beta_j\mid (i,j)\notin IJ\right\}>0$$

Thus, we define

$$\lambda_0=\min\left\{c_{i,j}-\alpha_i-\beta_j\mid (i,j)\notin IJ\wedge v_i\in W\wedge w_j\in W^c\right\}>0$$

Quality of the new dual solution

With $\lambda_0=\min\left\{c_{i,j}-\alpha_i-\beta_j\mid (i,j)\notin IJ\right\}>0$, we calculate

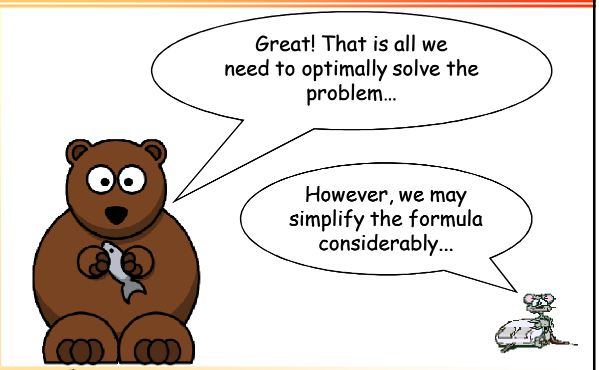
$$\sum_{i=1}^{m}a_i\cdot\left(\alpha_i+\lambda_0\cdot\tilde{\alpha}_i\right)+\sum_{j=1}^{n}b_j\cdot\left(\beta_j+\lambda_0\cdot\tilde{\beta}_j\right)=$$

$$\sum_{i=1}^{m}\left(a_i\cdot\alpha_i+\lambda_0\cdot a_i\cdot\tilde{\alpha}_i\right)+\sum_{j=1}^{n}\left(b_j\cdot\beta_j+\lambda_0\cdot b_j\cdot\tilde{\beta}_j\right)=$$

$$\sum_{i=1}^{m}a_i\cdot\alpha_i+\sum_{j=1}^{n}b_j\cdot\beta_j+\lambda_0\cdot\left(\sum_{i=1}^{m}a_i\cdot\tilde{\alpha}_i+\sum_{j=1}^{n}b_j\cdot\tilde{\beta}_j\right)=$$

$$\sum_{i=1}^{m}a_i\cdot\alpha_i+\sum_{j=1}^{n}b_j\cdot\beta_j+\lambda_0\cdot\left(\sum_{i=1}^{m}x_i^a\right)\underbrace{>}_{\text{If }\sum_{i=1}^{m}x_i^a>0}\sum_{i=1}^{m}a_i\cdot\alpha_i+\sum_{j=1}^{n}b_j\cdot\beta_j$$

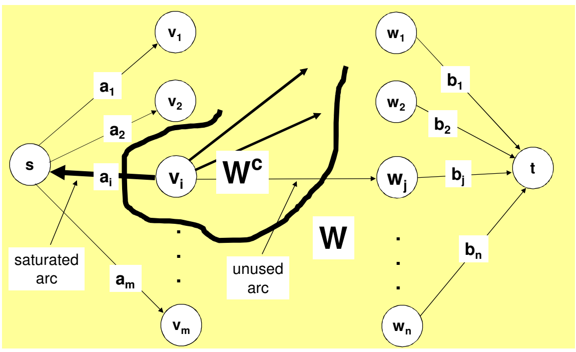
And what follows?



Important observation – Part 1

We consider the resulting constellation after applying the Max-Flow procedure. Addionally, we analyze the generated flow $x_{i,j}$. First of all, we consider arcs that vanish in the next iteration. This may happen only if $(i,j)\in IJ$ in the current iteration, but in the next one it holds $(i,j)\notin IJ$. This case is characterized that originally $\alpha_i+\beta_j=c_{i,j}$ applies, but subsequently $\hat{\alpha}_i+\hat{\beta}_j<c_{i,j}$ holds. Note that this is only possible if $\tilde{\alpha}_i+\tilde{\beta}_j<0 \Rightarrow \tilde{\alpha}_i+\tilde{\beta}_j=-1$. This is the constellation $v_i\in W^c \wedge w_j\in W$. It is illustrated on the next slide. Here, we directly conclude that the arc $(i,j)\in IJ$ was not used by the generated flow at all. Hence, we obtain $x_{i,j}=0$.

Illustration of this constellation



Consequence

- If we erase the edge (i,j) in the subsequent iteration, i.e., the solving of the modified (RP), this has no impact on the current flow $x_{i,j}$
- Note that the current flow does not make use of this arc
- Consequently, this arc is dispensable

Observations II

Now we consider arcs $(i,j) \in IJ$ with $x_{i,j} > 0$. We know that it holds $\hat{\alpha}_i + \hat{\beta}_j = c_{i,j} \Rightarrow \tilde{\alpha}_i + \tilde{\beta}_j = 0$. Therefore, the flow $x_{i,j} > 0$ can be kept on these arcs.

Anyhow, the resulting flow $x_{i,j}$ can be kept for the next iteration of solving (RP) that arises after updating α and β. Note that this update may cause additional arcs between the v_i- and w_j-nodes.

Calculating λ_0

$$\lambda_0 = \min\left\{c_{i,j} - \alpha_i - \beta_j \mid (i,j) \notin IJ \wedge v_i \in W \wedge w_j \in W^c\right\}$$

Thus, we can label all rows i in the reduced matrix $\left(c_{i,j} - \alpha_i - \beta_j\right)$ with $v_i \in W^c$. Additionally, we label all columns j with $w_j \in W$.
Then λ_0 is determined by the minimum unlabeled value.

We update $\left(c_{i,j} - \hat{\alpha}_i - \hat{\beta}_j\right)$ by applying the following rules:

Updating rules

We distinguish:

1. If (i,j) is unlabeled $\Rightarrow v_i \in W \wedge w_j \in W^c$

$\Rightarrow$ We subtract λ_0 from $c_{i,j} - \alpha_i - \beta_j$

2. If (i,j) is labeled twice $\Rightarrow v_i \in W^c \wedge w_j \in W$

$\Rightarrow \alpha_i + \beta_j = -1$. We add λ_0 to $c_{i,j} - \alpha_i - \beta_j$

3. If (i,j) is labeled only by the ith row or the jth column

$\Rightarrow (v_i \in W \wedge w_j \in W) \vee (v_i \in W^c \wedge w_j \in W^c) \Rightarrow \alpha_i + \beta_j = 0$

$c_{i,j} - \alpha_i - \beta_j$ is kept unchanged

Continuation of the example

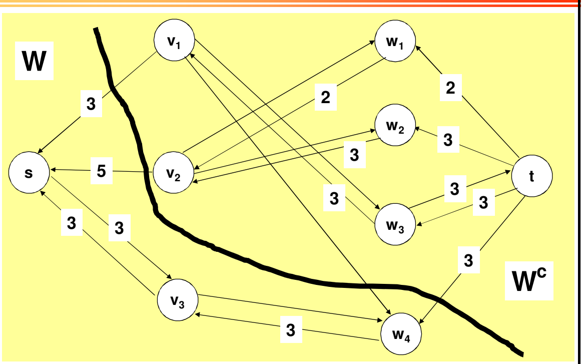
- Now, we resume our example which was introduced above
- Thus, first of all, we have to update the dual solution

With $\alpha = (0 \quad 0 \quad 1)^T \wedge \beta = (1 \quad 2 \quad 1 \quad 2)^T$

Reduced matrix is therefore : $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{pmatrix}$

$\Rightarrow IJ = \{(1,3),(1,4),(2,1),(2,2),(3,4)\}$

Illustration of the calculation



Updating the dual solution

$\Rightarrow$

$W = \{s, v_3, \boxed{w_4}\}$

$W^c = \{\boxed{v_1}, \boxed{v_2}, w_1, w_2, w_3, t\}$

With $\alpha = (0 \quad 0 \quad 1)^T \wedge \beta = (1 \quad 2 \quad 1 \quad 2)^T$

$$\left(c_{i,j} - \alpha_i - \beta_j\right) = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \cancel{2} & \cancel{1} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{1} & \cancel{1} \\ 2 & 2 & 4 & \cancel{0} \end{pmatrix}$$

Updating the dual solution

$$\lambda_0 = \min\{2,2,4\} = 2 \Rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 1 & 0 & 0+2 \\ 0 & 0 & 1 & 1+2 \\ 2-2 & 2-2 & 4-2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 0 \end{pmatrix} \Rightarrow$$

$\alpha^T = (0 \quad 0 \quad 1) \wedge \beta^T = (1 \quad 2 \quad 1 \quad 2)$

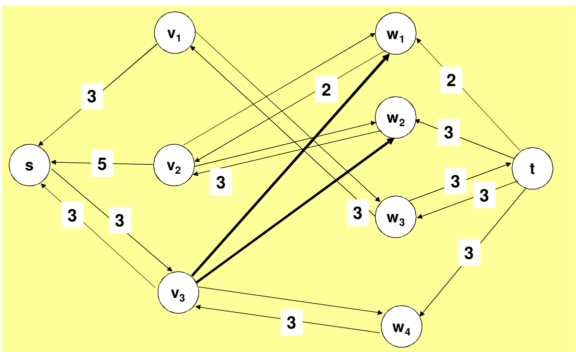
$\wedge \tilde{\alpha}^T = (0 \quad 0 \quad 1) \wedge \tilde{\beta}^T = (0 \quad 0 \quad 0 \quad -1)$

$\wedge \hat{\alpha}^T = (0 \quad 0 \quad 3) \wedge \hat{\beta}^T = (1 \quad 2 \quad 1 \quad 0)$

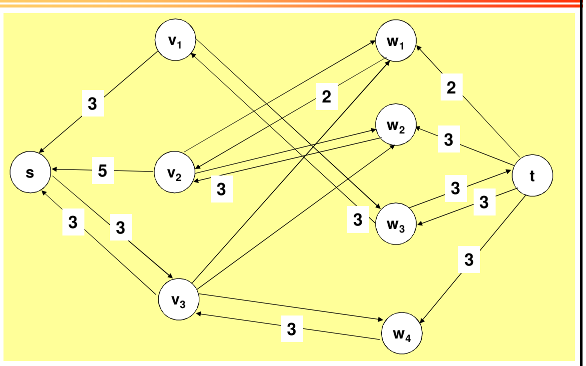
$\Rightarrow$ Thus, we get two new arcs $(3,1)$ and $(3,2)$ and lose one $(1,4)$.

$\Rightarrow IJ = \{(1,3),(2,1),(2,2),(3,1),(3,2),(3,4)\}$

Illustration



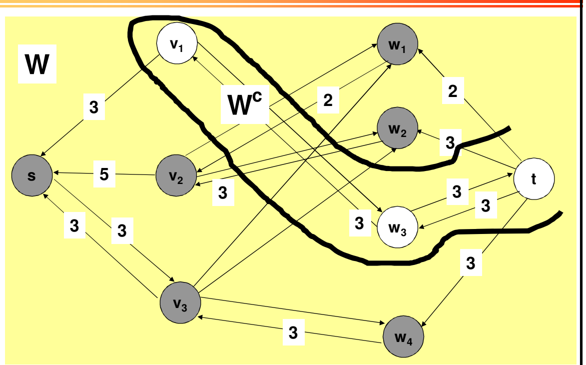
Applying Max-Flow



Results

- Unfortunately, we are not able to augment the flow
- Thus, *x* is kept as a maximum flow
- However, we have changed the sets *W* and W^c
- This is considered in the following

Applying Max-Flow



Updating the dual solution

$\Rightarrow$

$W=\{s,v_2,v_3,\boxed{w_1},\boxed{w_2},\boxed{w_4}\}$

$W^c=\{\boxed{v_1},w_3,t\}$

With $\alpha=\begin{pmatrix}0 & 0 & 3\end{pmatrix}^T \wedge \beta=\begin{pmatrix}1 & 2 & 1 & 0\end{pmatrix}^T$

$$\left(c_{i,j}-\alpha_i-\beta_j\right)=\begin{pmatrix}2 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\\ 0 & 0 & 2 & 0\end{pmatrix}\Rightarrow\begin{pmatrix}\cancel{2} & \cancel{1} & \cancel{0} & \cancel{2}\\ \cancel{0} & \cancel{0} & 1 & \cancel{3}\\ \cancel{0} & \cancel{0} & 2 & \cancel{0}\end{pmatrix}$$

Updating the dual solution

$$\lambda_0=\min\{2,1\}=1\Rightarrow\begin{pmatrix}2 & 1 & 0 & 2\\ 0 & 0 & 1 & 3\\ 0 & 0 & 2 & 0\end{pmatrix}\Rightarrow\begin{pmatrix}3 & 2 & 0 & 3\\ 0 & 0 & 0 & 3\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$\alpha^T=\begin{pmatrix}0 & 0 & 3\end{pmatrix}\wedge\beta^T=\begin{pmatrix}1 & 2 & 1 & 0\end{pmatrix}$

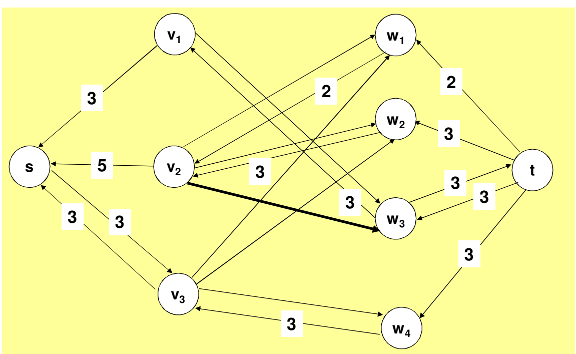
$\wedge\tilde{\alpha}^T=\begin{pmatrix}0 & 1 & 1\end{pmatrix}\wedge\tilde{\beta}^T=\begin{pmatrix}-1 & -1 & 0 & -1\end{pmatrix}$

$\Rightarrow\hat{\alpha}^T=\begin{pmatrix}0 & 1 & 4\end{pmatrix}\wedge\hat{\beta}^T=\begin{pmatrix}0 & 1 & 1 & -1\end{pmatrix}$

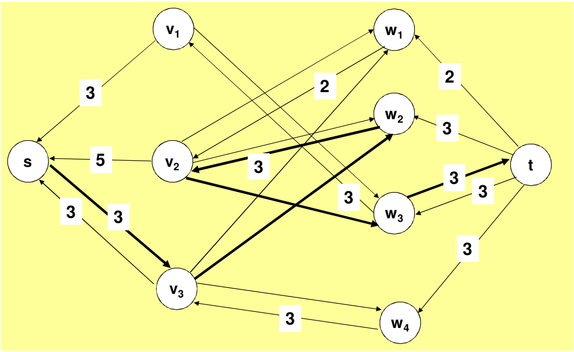
$\Rightarrow$ Thus, we get a new arcs $(2,3)$.

$\Rightarrow IJ=\{(1,3),(2,1),(2,2),(2,3)(3,1),(3,2),(3,4)\}$

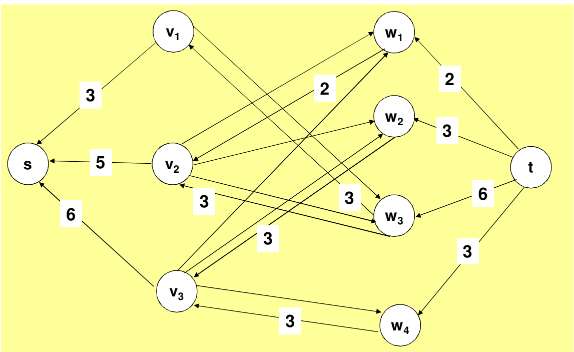
Modified network



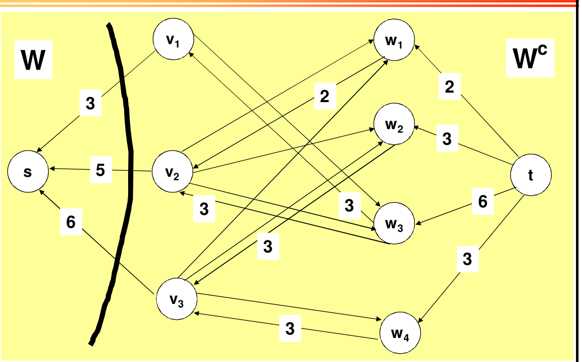
We obtain the augmented flow



Illustration



The new decomposition



The modified primal solution

$$\Rightarrow W = \{s,\} \wedge W^c = \{v_1, v_2, v_3, w_1, w_2, w_3, w_4, t\}$$

With $\alpha = (0 \quad 1 \quad 4)^T \wedge \beta = (0 \quad 1 \quad 1 \quad -1)^T$

$$x = \begin{pmatrix} 0 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix}$$

$\Rightarrow$ Is feasible for $a^T = (3 \quad 5 \quad 6) \wedge b^T = (2 \quad 3 \quad 6 \quad 3)$

Proof of optimality

$$\Rightarrow W = \{s,\} \wedge W^c = \{v_1, v_2, v_3, w_1, w_2, w_3, w_4, t\}$$

$\Rightarrow x_i^a = 0, \forall i \in \{1, \ldots, m+n\}$ and it holds:

$$c^T \cdot x = 1 \cdot 3 + 1 \cdot 2 + 3 \cdot 2 + 5 \cdot 3 + 3 \cdot 3 = 35$$

$$a^T \cdot \alpha + b^T \cdot \beta = 3 \cdot 0 + 5 \cdot 1 + 6 \cdot 4 + 2 \cdot 0 + 3 \cdot 1 + 6 \cdot 1 - 3 \cdot 1$$

$$= 5 + 24 + 3 + 6 - 3 = 38 - 3 = 35$$

$\Rightarrow x$ and (α, β) are optimal solutions!

Alpha-Beta-Algorithm

1. Construct a feasible dual solution to the TPP
   - Set $\beta_j = \min\{c_{ij} \mid i = 1, \ldots, m\}$ and $\alpha_i = \min\{c_{ij} - \beta_j \mid j = 1, \ldots, n\}$
   - Calculate the matrix with the reduced costs $\overline{c}_{ij} = c_{ij} - \alpha_i - \beta_j$
2. Prepare the network for the Max-Flow-Calculation
   - Nodes: $s, v_1, \ldots, v_m, w_1, \ldots, w_n, t$
   - Arcs: $\begin{matrix}(s, v_1), \ldots, (s, v_m) \\ (w_1, t), \ldots, (w_n, t)\end{matrix}$ with capacity $\begin{matrix}a_1, \ldots, a_m \\ b_1, \ldots, b_n\end{matrix}$
3. Furthermore: If and only if $\overline{c}_{ij} = 0$, the arc (v_i, w_j) exists with infinite capacity
4. Calculate the Maximum s-t-Flow in the network. Let W be the set of nodes reachable from node s in the corresponding s-t-Cut
5. While $W \neq \{s\}$, conduct the following steps (see next slide):

Alpha-Beta-Algorithm (Dual Solution Update)

- If $v_i \in W \Rightarrow \tilde{\alpha}_i = 1; v_i \in W^c \Rightarrow$, label the *i*-th row in the reduced cost matrix.
- If $w_j \in W \Rightarrow \tilde{\beta}_j = -1 \Rightarrow$, label the *j*-th column in the reduced cost matrix.
- All other variables of the DRP-solution $\tilde{\alpha}, \tilde{\beta}$ are set to 0.
- Set λ_0 to the minimum value of the unlabeled entries in the reduced cost matrix.
- Subtract λ_0 from every unlabeled entry and add it to every entry labeled twice in the reduced cost matrix.
- Set $\beta = \beta + \lambda_0 \tilde{\beta} \wedge \alpha = \alpha + \lambda_0 \tilde{\alpha}$
- Update the network as indicated by the new reduced cost matrix.
- Try to augment the current flow and update the set W.