9 Integer Programming

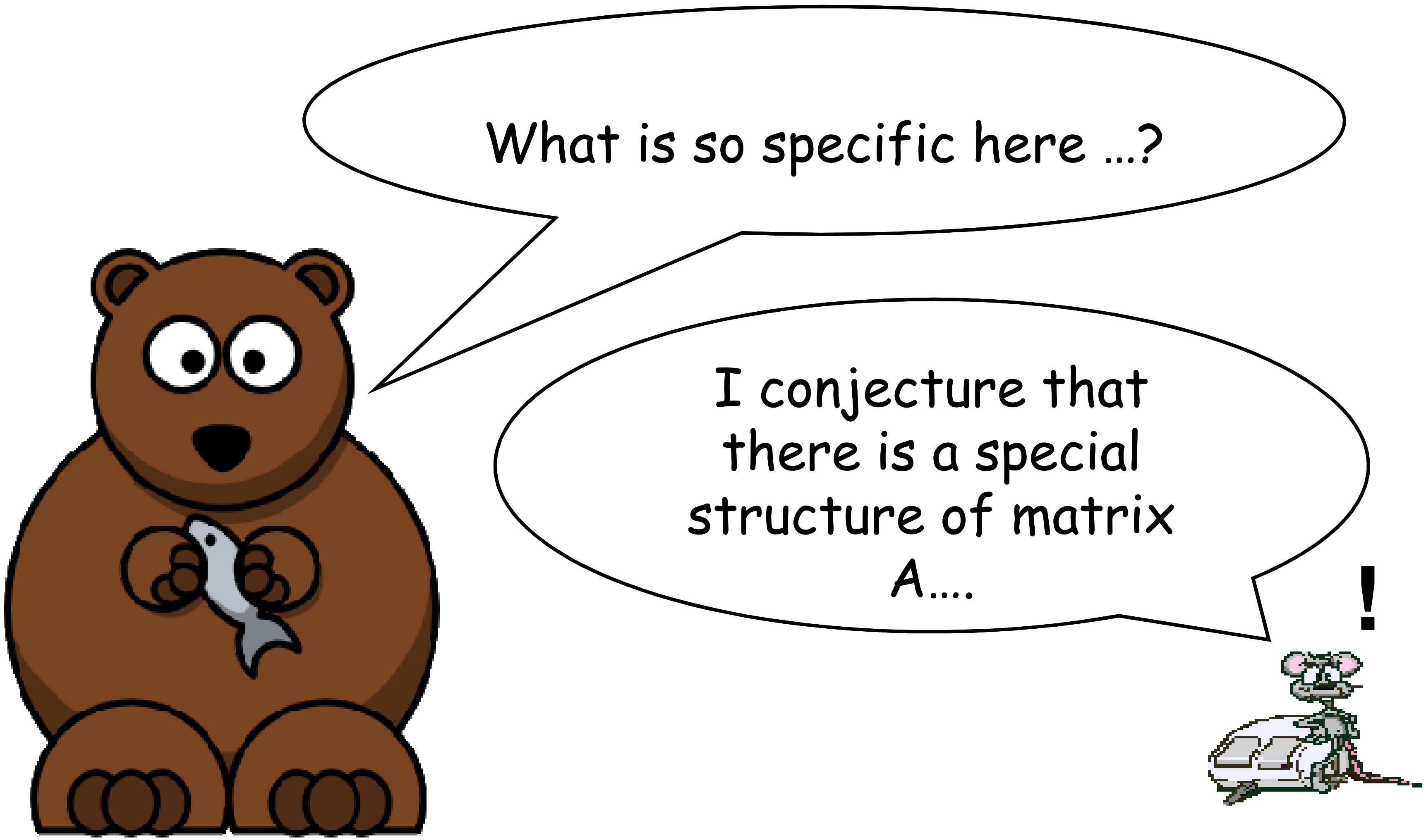
- In what follows, we consider a subset of Linear Programs where solutions, i.e., the variables as well as the parameters of the problem definition, are restricted to integers
- Although this leads to a considerable reduction of the size of the solution space, it complicates the solution process significantly
- It turns out that these problems cannot be solved efficiently, i.e., based on current knowledge, a solution of these problems cannot be guaranteed in polynomial time
- However, by inspecting specific problems introduced and analyzed above, it turns out that optimal solutions are already integer

9.1 Well-solvable problems

- Already introduced representatives of well-solvable problems are
  - Transportation Problem
  - Shortest Path Problem
  - Max-Flow

- The interesting question at this point is
  "WHY, i.e., what makes these problems such simple?"

And what follows?



Unimodular matrices

9.1.1 Definition

A matrix $B \in IR^{n \times n}$ is denoted as unimodular if and only if $|\det\ B| = 1$.

9.1.2 Definition

A matrix $B \in IR^{m \times n}$ is denoted as totally unimodular, in the following denoted as TUM, if and only if every square non-singular submatrix of A is unimodular.

We know that each singular square matrix A has a determinant equal to zero. Hence, we can conclude that a matrix $B \in IR^{m \times n}$ is denoted as totally unimodular if and only if every square submatrix A has a determinant equal to -1,0,+1.

Examples

- Let us consider some examples

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \text{ since } det\left(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\right) = +1\cdot 1-(-1)\cdot 1 = 1+1 = 2$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ since } det\left(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right) = 1\cdot(1-0)-1\cdot(0-1)+0 = 2$$

- However, consider the zero matrix
  - Obviously, it is NOT unimodular since the determinant has the value zero
  - However, there is no non-singular sub-matrix. Thus, nothing to fulfill wherefore the matrix is TUM

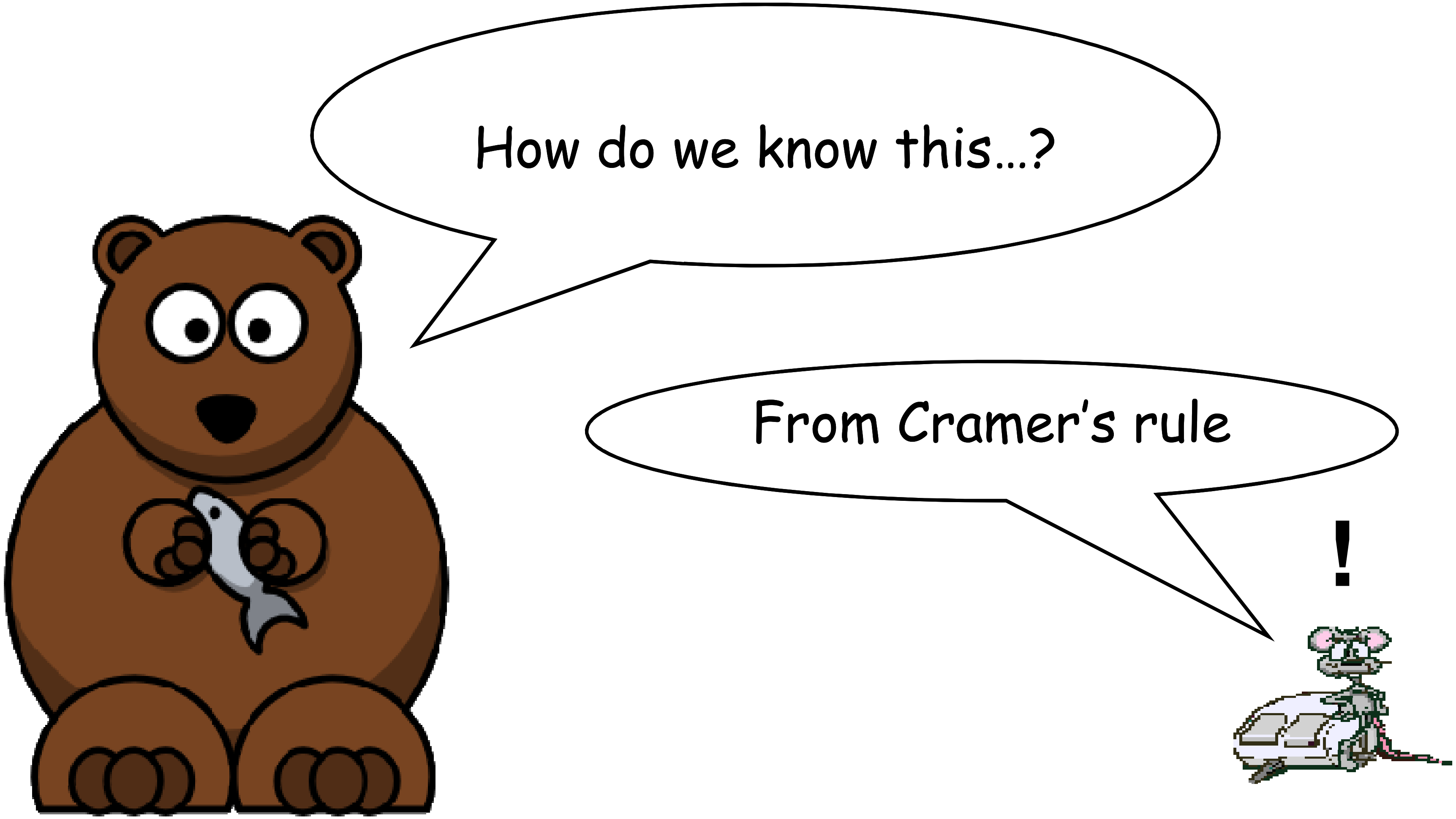
Effect of unimodularity

- Consider the LP

$$\boxed{\text{Min } c^T \cdot x, \text{ s.t. } A \cdot x = b \wedge x \geq 0}$$

- Furthermore, according to a basis *B,* let matrix A_B be unimodular
- Then, we can conclude that the corresponding basic feasible solution (bfs) is an integer solution

And what follows?



Cramer's rule

- Consider the adjoint matrix

$$\boxed{adj\left(A_B\right)_{i,j} = (-1)^{i+j} \cdot det\left(A_B\left(i \mid j\right)\right)}$$

- Note that $A_B(i|j)$ arises from A_B by erasing the ith row and jth column
- Then, we know that $$\boxed{A_B^{-1} = \frac{1}{det\left(A_B\right)} \cdot adj\left(A_B\right)}$$

- Since the entries of the adjoint matrix are obviously integers, the inverted matrix has only integer entries

The basic feasible solution

- Thus, we get

$$\boxed{\left(x_B, x_N\right) = \left(A_B^{-1} \cdot b, 0\right) = \left(\frac{1}{det\left(A_B\right)} \cdot adj\left(A_B\right) \cdot b, 0\right)}$$

  as feasible integer solution

- Consequently, we can conclude the following Theorem

Main consequence

9.1.3 Theorem

A linear program Min $c^T \cdot x$, s.t. $A \cdot x = b$ with a totally unimodular matrix A has only integer basic feasible solutions.

This is also true for problems Min $c^T \cdot x$, s.t. $A \cdot x \geq b$
and Max $c^T \cdot x$, s.t. $A \cdot x \leq b$.

Proof of Theorem 9.1.3

- The Theorem follows immediately out of the following simple observations
  - Owing to unimodularity, each basic feasible solution becomes integer
  - If we have a totally unimodular matrix A the combined matrixes (E,A) and (-E,A) are also totally unimodular
  - Thus, we always obtain basic feasible solutions comprising only integer values
- In what follows, we are looking for simple criteria that guarantee unimodularity for a given matrix

Criteria for unimodularity

9.1.4 Proposition

A matrix A is totally unimodular if

- Matrix A has only -1, 0, +1 entries
- Each column comprises at most two non-zero elements
- The rows of A can be partitioned into two subsets A_1 and A_2 (i.e., $A_1 \cup A_2 = \{1,\ldots,m\}$) such that two non-zero elements in a column are either in the same set of rows if they have different signs or they are in different sets of rows if they have equal signs

Proof of Proposition 9.1.4

- We identify an arbitrary square submatrix B of the matrix A
- Obviously, the given criteria also apply to this submatrix
- We show that det(B)={0,-1,1} by induction by the size n of the submatrix B
- We commence with n=1: Here, the proposition is obviously true
- Let us assume that the determinant of all submatrices with size lower than n have value {0,-1,1}
- Now, we distinguish three cases
  - Case 1: B has a zero column. Obviously, by generating the determinant by this column, we obtain det(B)=0
  - Case 2: B has a column with one value equal to 1 or -1. Then, by generating the determinant by this column, we know that det(B)=det(C) or det(B)=-det(C)

Proof of Proposition 9.1.4

  - Case 3: All columns have exactly two values unequal to zero. Then, the sets A_1 and A_2 provide us with a separation. Specifically, we have

$$\boxed{\sum_{i\in A_1} a_{i,j} = \sum_{i\in A_2} a_{i,j}, \forall j \in \{1,\ldots,n\}}$$

  - I.e., the matrix is obviously singular and, therefore, we have det(B)=0

- This completes the proof

Direct consequences

- **Transportation Problem**
  - What kind of matrix is it?

$$(P)\ \text{Minimize } c^T \cdot x$$

$$\text{s.t.} \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \dots & \dots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} \cdot x = \begin{pmatrix} a_1 \\ \dots \\ \dots \\ a_m \\ b \end{pmatrix}$$

$$x = \left(x_{1,1},\dots,x_{1,n},\dots,x_{m,1},\dots,x_{m,n}\right)^T \geq 0$$

  - Obviously, we have exactly two 1 values and nothing else in each column
  - Moreover, we have a separation of this matrix
  - Specifically, on the one side A_1={1,...,m} and on the other side A_2={m+1,…,m+n}. Hence, by applying Proposition 8.1.4, we know that A is totally unimodular

Direct consequences

- **Vertex-arc adjacency matrix**
  - What kind of matrix is it?

$$A=\left(\alpha_{i,k}\right)_{1\le i\le n;1\le k\le m}\text{, with }\alpha_{i,k}=\begin{cases}+1 \text{ when } \exists j\in V: e_k=(i,j)\\ -1 \text{ when } \exists j\in V: e_k=(j,i)\\ \quad 0 \text{ otherwise}\end{cases}$$

$$\alpha_{i,k}=1\Rightarrow i \text{ is source of arc } e_k;\quad \alpha_{i,k}=-1\Rightarrow i \text{ is sink of arc } e_k$$

  - Obviously, we have exactly one “1-value” and one “-1-value” in each column
  - Moreover, we have a trivial separation of this matrix
  - Specifically, on the one side A_1={1,...,n} comprises all rows of matrix A and on the other side A_2 is empty. Hence, by applying Proposition 8.1.4, we know that A is totally unimodular

Criteria for unimodularity

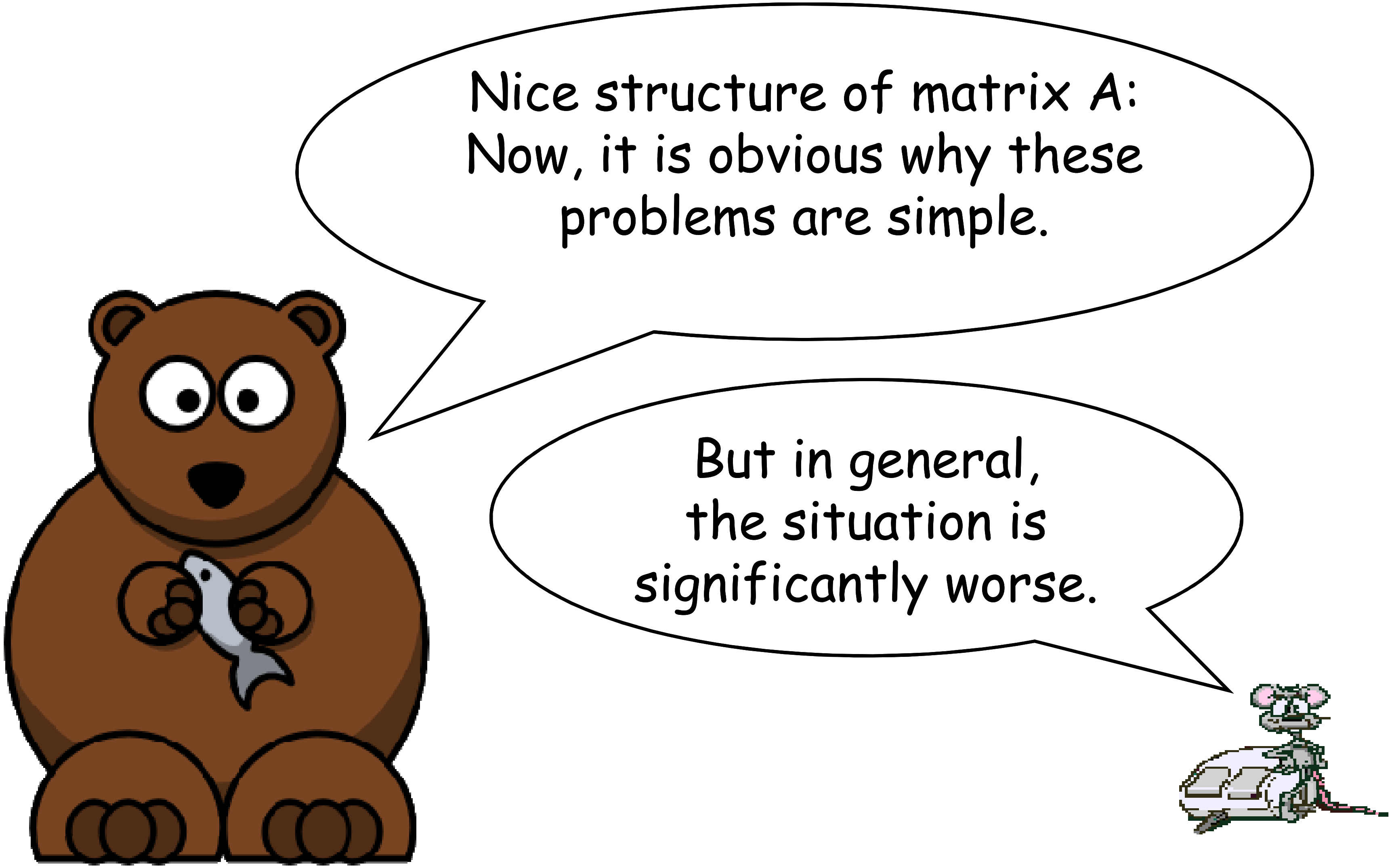
9.1.5 Corollary

A matrix A is totally unimodular if and only if

- the transpose matrix A^T is totally unimodular
- the matrix (A,E) is totally unimodular

The Proof follows directly out of Proposition 9.1.4

And what follows?



In general …

- Linear Integer Programs are unfortunately NP hard
- I.e., out of current knowledge, we assume that it is not possible to solve this problem with an algorithm whose running time is polynomially bounded
- Unfortunately, since those problems are of significant interest, we have to provide new techniques
  - that find best integer solutions
  - but cannot avoid exponential running times for specific worst case scenarios
- This is addressed in the following sections

9.2 Cutting Plane Method

- The basic idea goes back to Gomory (1958)
- By optimally solving the continuous problem (i.e., the so-called **LP-relaxation**), we may face two different constellations
  - The found solution is already integer, i.e., an optimal solution is also found for the integer variant of the continuous problem
  - Otherwise, the found optimal solution comprises some entries that are not integers
- The second case is handled as follows
  - Integrate an additional restriction that excludes the optimal non-integer solution, but
  - keeps all integer solutions

We consider an example

$$\begin{aligned}&\text{Maximize } 1\cdot x_1+1\cdot x_2\\&\text{s.t. } -6\cdot x_1+8\cdot x_2\le 3\\&\qquad 2\cdot x_1-2\cdot x_2\le 1\\&\qquad x_1,x_2\ge 0\\&\qquad x_1,x_2 \text{ are integers}\end{aligned}$$

- Therefore, we obtain for the LP-relaxation

$$\begin{array}{c|cccc}0&-1&-1&0&0\\\hline 3&-6&8&1&0\\1&2&-2&0&1\end{array}\Rightarrow\begin{array}{c|cccc}0&[-1]&-1&0&0\\\hline 3&-6&8&1&0\\1&(2)&-2&0&1\end{array}\Rightarrow\begin{array}{c|cccc}0&-1&-1&0&0\\\hline 3&-6&8&1&0\\ \frac{1}{2}&1&-1&0&\frac{1}{2}\end{array}$$

$$\Rightarrow\begin{array}{c|cccc}\frac{1}{2}&0&-2&0&\frac{1}{2}\\\hline 3&-6&8&1&0\\ \frac{1}{2}&1&-1&0&\frac{1}{2}\end{array}\Rightarrow\begin{array}{c|cccc}\frac{1}{2}&0&[-2]&0&\frac{1}{2}\\\hline 6&0&(2)&1&3\\ \frac{1}{2}&1&-1&0&\frac{1}{2}\end{array}\Rightarrow\begin{array}{c|cccc}\frac{1}{2}&0&-2&0&\frac{1}{2}\\\hline 3&0&1&\frac{1}{2}&\frac{3}{2}\\ \frac{1}{2}&1&-1&0&\frac{1}{2}\end{array}$$

And obtain finally

- as follows

$$\begin{array}{c|cccc}
\frac{1}{2} & 0 & -2 & 0 & \frac{1}{2} \\ \hline
3 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \\
\frac{1}{2} & 1 & -1 & 0 & \frac{1}{2}
\end{array}
\Rightarrow
\begin{array}{c|cccc}
\frac{1}{2} & 0 & -2 & 0 & \frac{1}{2} \\ \hline
3 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \\
\frac{7}{2} & 1 & 0 & \frac{1}{2} & 2
\end{array}
\Rightarrow
\begin{array}{c|cccc}
\frac{13}{2} & 0 & 0 & 1 & \frac{7}{2} \\ \hline
3 & 0 & 1 & \frac{1}{2} & \frac{3}{2} \\
\frac{7}{2} & 1 & 0 & \frac{1}{2} & 2
\end{array}$$

- We obtain the solution x=(3,7/2)
- Obviously, this solution is not integer

Let us consider the final tableau

- It holds:
- First row $z = z_0 + \overline{c}_N^T \cdot x_N$
- The rest $\overline{b} = x_B + \overline{A}_N \cdot x_N = x_B + A_B^{-1} \cdot A_N \cdot x_N$
- By setting

$$y = \left(y_{i,j}\right)_{0 \le i \le m; 0 \le j \le n} = \begin{pmatrix} -z_0 & \overline{c}^T \\ \overline{b} & \overline{A} \end{pmatrix} \text{ and } x_{B(0)} = -z_0$$

- we may write restriction (1)

$$y_{i,0} = x_{B(i)} + \sum_{j \in N} y_{i,j} \cdot x_j, \forall i \in \{0,\ldots,m\} \qquad (1)$$

- I.e., the left-hand side always represents a combination of a basic variable and non-basic variables
- It is fulfilled by all feasible solutions to the LP

Conclusions

- Since we know $x_N \geq 0$, we conclude

$$\boxed{y_{i,0} \geq x_{B(i)} + \sum_{j \in N} \lfloor y_{i,j} \rfloor \cdot x_j, \forall i \in \{0,\ldots,m\}}$$

- Let us now assume that we have an integer solution, i.e., x and z are integer vectors
  - In that case, the left-hand side becomes integer, i.e., we have only summation and multiplication operations with integers
  - Thus, we directly obtain as restriction (2)

$$\boxed{\lfloor y_{i,0} \rfloor \geq x_{B(i)} + \sum_{j \in N} \lfloor y_{i,j} \rfloor \cdot x_j, \forall i \in \{0,\ldots,m\} \qquad (2)}$$

Observation

- While (1) applies to all feasible solutions, (2) is fulfilled only if x_B is integer
- Note that this follows directly from the fact that

$$\underbrace{\lfloor y_{i,0} \rfloor \geq x_{B(i)} + \underbrace{\sum_{j \in N} \lfloor y_{i,j} \rfloor \cdot \underbrace{x_j}_{=0}}_{=0}}_{=x_{B(i)}=y_{i,0}}, \forall i \in \{0,\ldots,m\} \qquad (2)$$

- And if $x_{B(i)}$ is not integer, we obtain

$$\underbrace{\underbrace{\lfloor y_{i,0} \rfloor}_{<y_{i,0}=x_{B(i)}} \geq \underbrace{x_{B(i)} + \underbrace{\sum_{j \in N} \lfloor y_{i,j} \rfloor \cdot \underbrace{x_j}_{=0}}_{=0}}_{=x_{B(i)}=y_{i,0}}}_{\Rightarrow \lfloor y_{i,0} \rfloor < x_{B(i)}}, \forall i \in \{0,\ldots,m\}$$

Generating a new restriction

- In order to obtain the desired new restriction, we have to get rid of $x_{B(i)}$. We just subtract (1) from (2)

$$y_{i,0} = x_{B(i)} + \sum_{j \in N} y_{i,j} \cdot x_j, \forall i \in \{0,...,m\} \qquad (1)$$

$$\lfloor y_{i,0} \rfloor \geq x_{B(i)} + \sum_{j \in N} \lfloor y_{i,j} \rfloor \cdot x_j, \forall i \in \{0,...,m\} \qquad (2)$$

$$\Rightarrow \lfloor y_{i,0} \rfloor - y_{i,0} \geq \sum_{j \in N} \left(\lfloor y_{i,j} \rfloor - y_{i,j} \right) \cdot x_j \qquad (2)-(1)$$

$$\boxed{\lfloor y_{i,0} \rfloor - y_{i,0} = \sum_{j \in N} \left(\lfloor y_{i,j} \rfloor - y_{i,j} \right) \cdot x_j + x_{n+1} \quad \text{with } x_{n+1} \text{ as a new slack variable}}$$

- Adding the last restriction (**cut**) to the Simplex tableau, we exclude the fractional solution x_B but do not loose any integer solution. In fact, the restriction is designed such that **at least one integer solution is on its hyperplane**
- IPs are still difficult! We don't know how many cuts to add

Resume with our example

$$\begin{array}{c|cccc} 13/2 & 0 & 0 & 1 & 7/2 \\ \hline 3 & 0 & 1 & 1/2 & 3/2 \\ 7/2 & 1 & 0 & 1/2 & 2 \end{array} \Rightarrow \begin{array}{c|ccccc} 13/2 & 0 & 0 & 1 & 7/2 & 0 \\ \hline 3 & 0 & 1 & 1/2 & 3/2 & 0 \\ 7/2 & 1 & 0 & 1/2 & 2 & 0 \\ -1/2 & 0 & 0 & 0 & -1/2 & 1 \end{array}$$

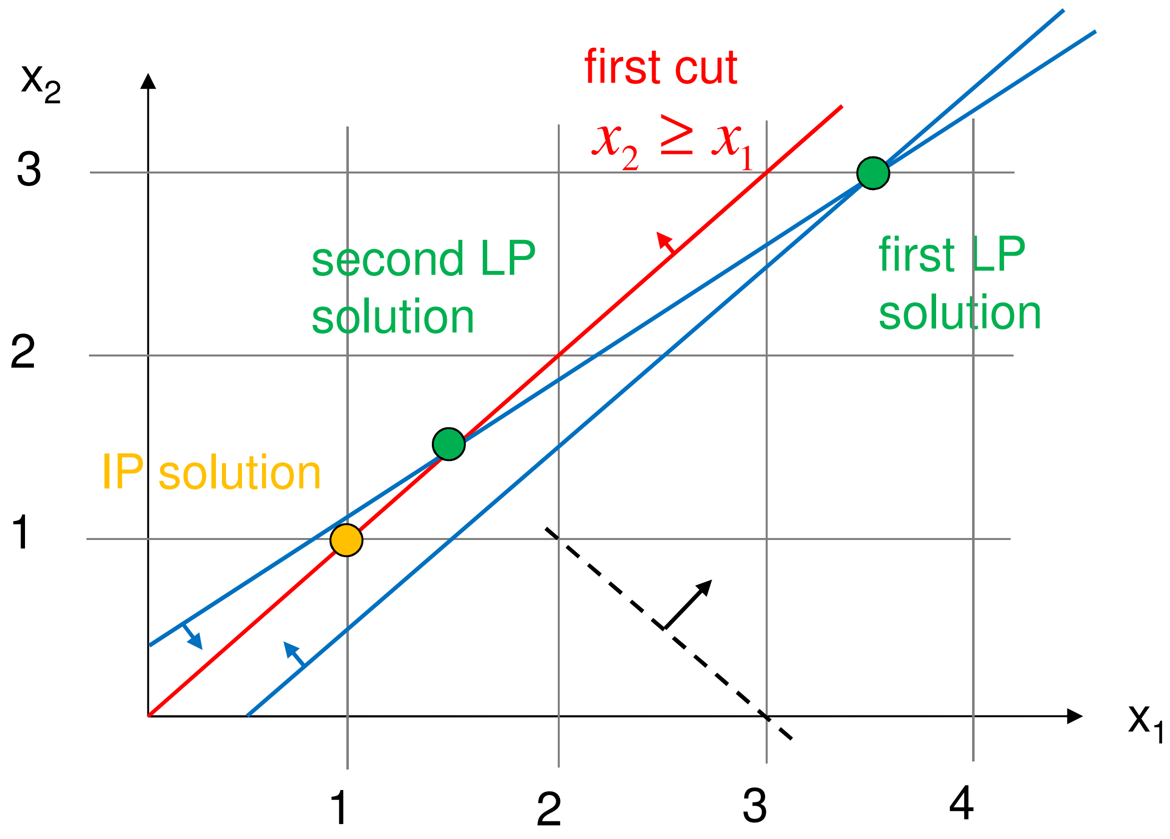
- Note that the first row has led to the **first cut**

$$\lfloor y_{0,0} \rfloor - y_{0,0} = \sum_{j \in N} \left(\lfloor y_{0,j} \rfloor - y_{0,j} \right) \cdot x_j + x_5 \Rightarrow \left\lfloor \frac{13}{2} \right\rfloor - \frac{13}{2} = (1-1) \cdot x_3 + \left(\left\lfloor \frac{7}{2} \right\rfloor - \frac{7}{2} \right) \cdot x_4 + x_5$$

$$-\frac{1}{2} = -\frac{1}{2} \cdot x_4 + x_5$$

- Obviously, the resulting solution is not feasible since $x_5<0$
- However, owing to the fact that we introduce an additional dual variable, the dual solution obviously stays feasible
- Hence, we apply the Dual Simplex Algorithm

First cut



$$-\tfrac{1}{2} \geq -\tfrac{1}{2} \cdot x_4 = -\tfrac{1}{2} \underbrace{\left(1 - 2 \cdot x_1 + 2 \cdot x_2\right)}_{\text{2nd restriction}} \Leftrightarrow x_2 \geq x_1$$

Applying the Dual Simplex Algorithm

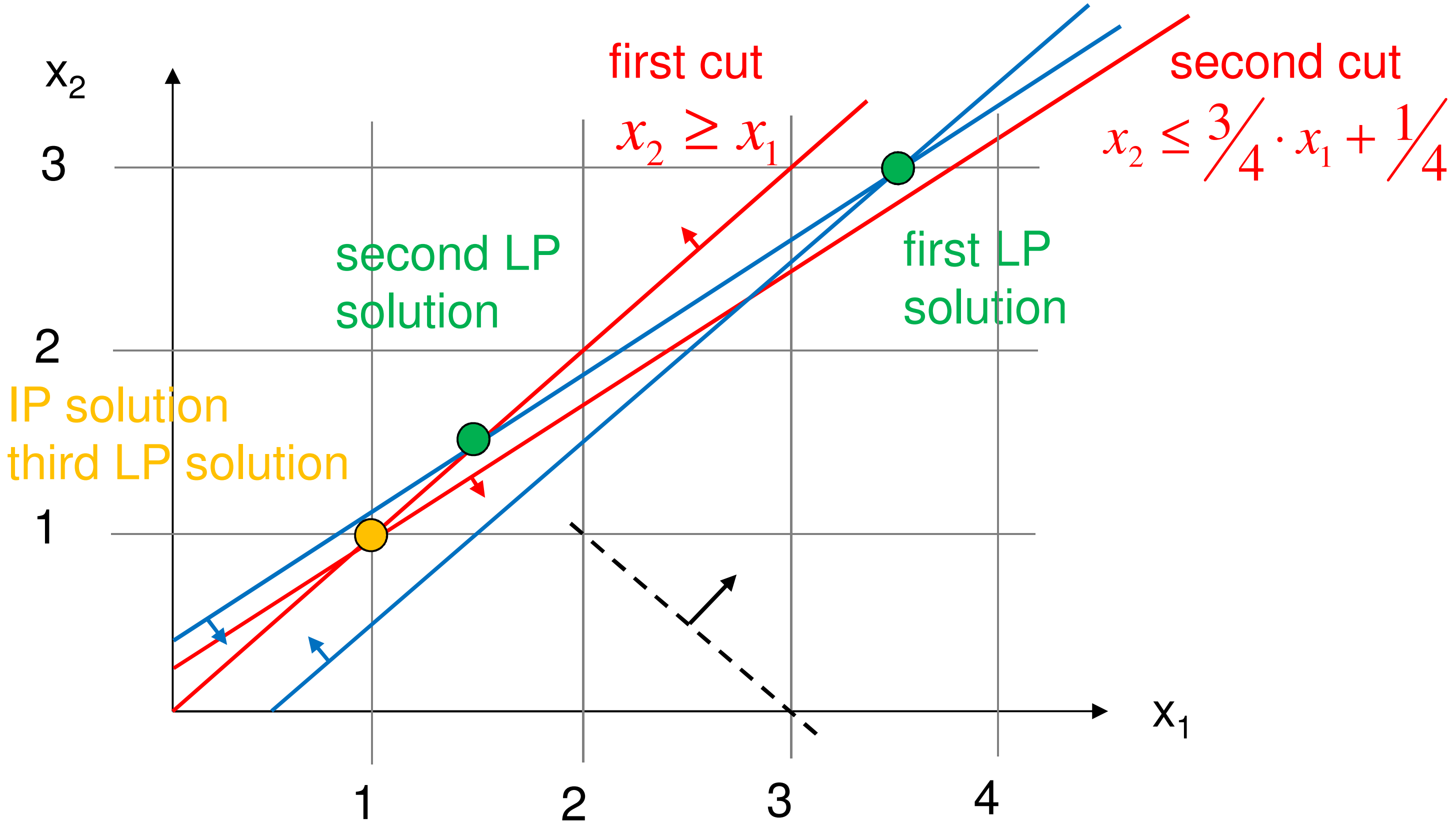
$$
\begin{array}{c|ccccc}
13/2 & 0 & 0 & 1 & 7/2 & 0 \\ \hline
3 & 0 & 1 & 1/2 & 3/2 & 0 \\
7/2 & 1 & 0 & 1/2 & 2 & 0 \\
\left(-1/2\right) & 0 & 0 & 0 & \left[-1/2\right] & 1
\end{array}
\Rightarrow
\begin{array}{c|ccccc}
3 & 0 & 0 & 1 & 0 & 7 \\ \hline
3/2 & 0 & 1 & 1/2 & 0 & 3 \\
3/2 & 1 & 0 & 1/2 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 & -2
\end{array}
$$

- We obtain the second optimal LP solution $x^T = (3/2, 3/2, 0, 1, 0)$
- This solution is not integer and we introduce a **second cut:**

$$\lfloor y_{1,0} \rfloor - y_{1,0} = \sum_{j \in N} \left(\lfloor y_{1,j} \rfloor - y_{1,j} \right) \cdot x_j + x_6 \Rightarrow \left\lfloor \frac{3}{2} \right\rfloor - \frac{3}{2} = \left(\left\lfloor \frac{1}{2} \right\rfloor - \frac{1}{2} \right) \cdot x_3 + (3-3) \cdot x_5 + x_6$$

$$-\frac{1}{2} = -\frac{1}{2} \cdot x_3 + x_6$$

Second cut



$$-\tfrac{1}{2} \geq -\tfrac{1}{2} \cdot x_3 = -\tfrac{1}{2}\underbrace{\left(3 - 8 \cdot x_2 + 6 \cdot x_1\right)}_{\text{1st restriction}} \Leftrightarrow x_2 \leq \tfrac{3}{4} \cdot x_1 + \tfrac{1}{4}$$

Additional constraint

$$
\begin{array}{c|cccccc}
3 & 0 & 0 & 1 & 0 & 7 \\
\hline
3/2 & 0 & 1 & 1/2 & 0 & 3 \\
3/2 & 1 & 0 & 1/2 & 0 & 4 \\
1 & 0 & 0 & 0 & 1 & -2
\end{array}
\Rightarrow
\begin{array}{c|cccccc}
3 & 0 & 0 & 1 & 0 & 7 & 0 \\
\hline
3/2 & 0 & 1 & 1/2 & 0 & 3 & 0 \\
3/2 & 1 & 0 & 1/2 & 0 & 4 & 0 \\
1 & 0 & 0 & 0 & 1 & -2 & 0 \\
\left(-1/2\right) & 0 & 0 & \left[-1/2\right] & 0 & 0 & 1
\end{array}
\Rightarrow
\begin{array}{c|cccccc}
2 & 0 & 0 & 0 & 0 & 7 & 2 \\
\hline
1 & 0 & 1 & 0 & 0 & 3 & 1 \\
1 & 1 & 0 & 0 & 0 & 4 & 1 \\
1 & 0 & 0 & 0 & 1 & -2 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & -2
\end{array}
$$

- We obtain the third optimal LP solution x^T=(1,1,1,1,0,0)
- Thus, we obtain the optimal IP solution x^T=(1,1)

Gomory's Cutting Plane Method

1. Solve the LP relaxation with the Simplex Algorithm to optimality.

   Let α^j be the jth column with $j = 0,1,\ldots,n$ of the optimal tableau and

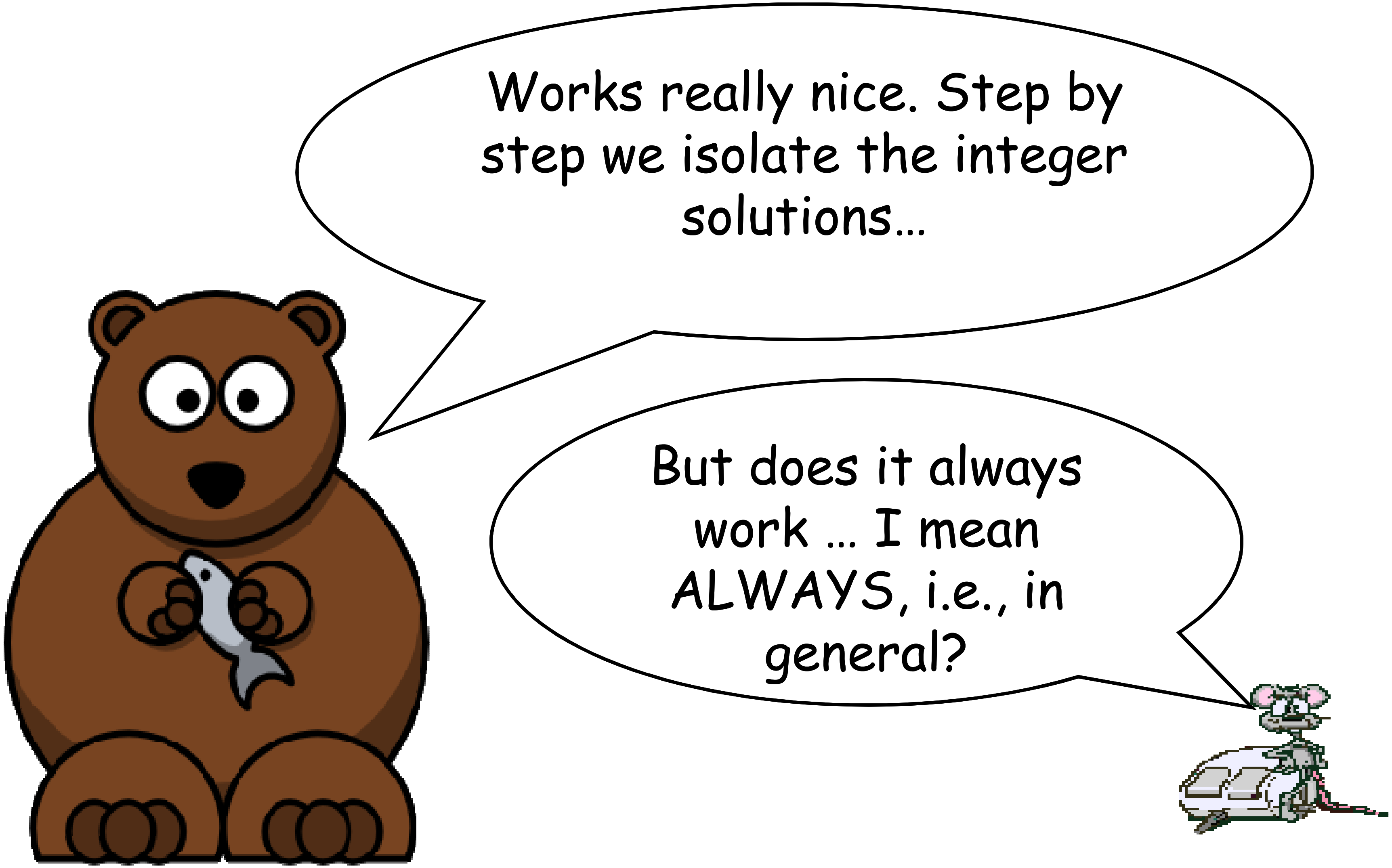
   hence, $\alpha^0 = \left(\bar{z}_0, \bar{b}_1, \ldots, \bar{b}_m\right)^T \wedge \alpha^j = \left(\bar{c}_j, \bar{a}_1^j, \ldots, \bar{a}_m^j\right)^T$, $j = 1,\ldots,n$.
2. If the LP solution space is unbounded, terminate since the ILP is unbounded
3. If $\alpha^0 \in \mathbb{Z}^{m+1}$, terminate since the integer solution is optimal to the ILP.
4. Select the row with the smallest index i_0 with $\alpha_{i_0}^0 \notin \mathbb{Z}$ and

   add the following Gomory cut to the optimal tableau:

   $$\left\lfloor \alpha_{i_0}^0 \right\rfloor - \alpha_{i_0}^0 = \sum_{j \in N} \left(\left\lfloor \alpha_{i_0}^j \right\rfloor - \alpha_{i_0}^j \right) \cdot x_j + x_{n+1}$$
5. Apply the lexicographic version of the Dual Simplex Algorithm.
6. Go to 2.

Note that the lexicographic version of the Dual Simplex Algorithm prevents **cycling**!

Finiteness of the algorithm



Finiteness of the algorithm

- In what follows, we consider the question whether the algorithm will always terminate if the original problem has an finite upper bound
- Therefore, in order to provide an understandable structure of pivoting, we first introduce the so-called lexicographic order
- This order allows us to attain significant insight into the structure of the resulting tableaus after each iteration

Lexicographic order

9.2.1 Definition (lexicographically positive)

$x \in IR^n$ is denoted as lexicographically positive if and only if the lowest numbered non-zero entry of x is positive. I.e., if and only if it holds: $x_{min\{i|x_i \neq 0\}} > 0$. If it holds that $x = 0$, we say x is lex-zero.

9.2.2 Definition (lexicographical order)

$x \in IR^n$ has an earlier position than $y \in IR^n$ in the lexicographical order if and only if $x - y \in IR^n$ is lexicographically positive. We write $x >^L y$.

Examples

It holds that:

$$(0,0,1,0) >^L (0,0,0,2)$$

$$(1,0,0,0) >^L (0,9,5,2)$$

$$(-2,0,0,0) <^L (-1,9,5,2)$$

$$(1,3,7,2) <^L (1,3,7,2,0,9,5,2)$$

$$(1,3,7,2) >^L (1,3,7,2,0,-9,5,2)$$

Consequences

- $>^L$ is obviously a complete ordering of the elements in IR^n
- Now, we have to define how the lexicographical version of the Dual Simplex Algorithm works in detail
- In this procedure, in order to break ties, the largest lexicographical column is always taken to improve the current dual solution

The Lexicographical Dual Simplex

9.2.3 Theorem

We consider the Simplex tableau defined by

$$\begin{array}{c|cc} -z_0 & 0 & \overline{c}_N^T \\ \hline \overline{b} & E_{(B)} & \overline{A}_N \end{array}, \text{with } \overline{c}_N^T \geq 0 \wedge \exists i{:}\overline{b}_i < 0$$

Thus, we may apply the Dual Simplex Algorithm. Moreover, $\alpha^0, \alpha^1, ..., \alpha^n$ are the columns of the tableau. We assume that all these columns (starting with column 1), i.e., the columns $\alpha^1, ..., \alpha^n$, are lexicographically positive $\left(\text{if not, we introduce an additional restriction } 1^T \cdot x + x_{n+1} \leq M\right)$.

Then, the Dual Simplex Algorithm terminates after conducting a finite number of steps complying with the following rules

1. Select an arbitrary i_0 fulfilling $a_{i_0}^0 < 0$
2. Determine t by $\dfrac{\alpha^t}{-a_{i_0}^t} = lex - \min\limits_j \left\{ \dfrac{\alpha^j}{-a_{i_0}^j} \mid a_{i_0}^j < 0 \right\}$

Proof of Theorem 9.2.3

- During the execution of each application of the dual simplex it holds that
  - All columns 1,…,n stay lex-positive throughout the computation
  - Column zero strictly lex-decreases
- This results from the following facts

$\alpha_i \left(1 \le i \le n\right)$ stays lex-positive after pivoting. The i_0th row becomes

$\tilde{\alpha}_{i_0} = \dfrac{\alpha_{i_0}}{\alpha_{i_0}^t}$, with $\alpha_{i_0}^t < 0 \Rightarrow \tilde{\alpha}_t$ is lex-positive since α_t is lex negative

$\left(\alpha_{i_0}^0 < 0\right)$.

The column t $\left(= \tilde{\alpha}^t\right)$ becomes $\left(0,\ldots,0,1,0,\ldots,0\right)^T$.

Proof of Theorem 9.2.3

We consider the rth column $(r \neq t)$ and compute $\tilde{\alpha}_i^r = \alpha_i^r - \frac{\alpha_i^t \cdot \alpha_{i_0}^r}{\alpha_{i_0}^t} = \alpha_{i_0}^r \cdot \left(\frac{\alpha_i^r}{\alpha_{i_0}^r} - \frac{\alpha_i^t}{\alpha_{i_0}^t} \right)$

We consider the first non-zero element $\max\{\alpha_i^r, \alpha_i^t\}$. Since both columns are lex-positive,

we have at this lowest numbered row i: $\alpha_i^r \geq 0, \alpha_i^t \geq 0$. We additionally assume that $\alpha_{i_0}^r > 0$.

Due to $\alpha_{i_0}^t < 0$, we conclude $\tilde{\alpha}_i^r > 0$ and $\tilde{\alpha}^r$ is lex-positive. Now, we assume $\alpha_{i_0}^r < 0$.

Due to the choice of column t, we know that the column with the entry

$\left(\frac{\alpha_i^r}{\alpha_{i_0}^r} - \frac{\alpha_i^t}{\alpha_{i_0}^t} \right)$ at row i is lex-positive since the first non-zero element j coincides

with $\left(\frac{\alpha_j^r}{\alpha_{i_0}^r} - \frac{\alpha_j^t}{\alpha_{i_0}^t} \right) = \left(\frac{\alpha_j^t}{-\alpha_{i_0}^t} - \frac{\alpha_j^r}{-\alpha_{i_0}^r} \right) < \left(\frac{\alpha_j^t}{-\alpha_{i_0}^t} - \frac{\alpha_j^t}{-\alpha_{i_0}^t} \right) = 0$ and we have $\alpha_{i_0}^r < 0$.

Consequently, we obtain for the first non-zero position: $\tilde{\alpha}_j^r = \alpha_{i_0}^r \cdot \left(\frac{\alpha_j^r}{\alpha_{i_0}^r} - \frac{\alpha_j^t}{\alpha_{i_0}^t} \right) > 0.$

Proof of Theorem 9.2.3

We consider the column zero and compute

$\tilde{\alpha}_i^0 = \alpha_i^0 - \dfrac{\alpha_i^t \cdot \alpha_{i_0}^0}{\alpha_{i_0}^t}$. We know that $\alpha_{i_0}^0 < 0$ and $\alpha_{i_0}^t < 0$.

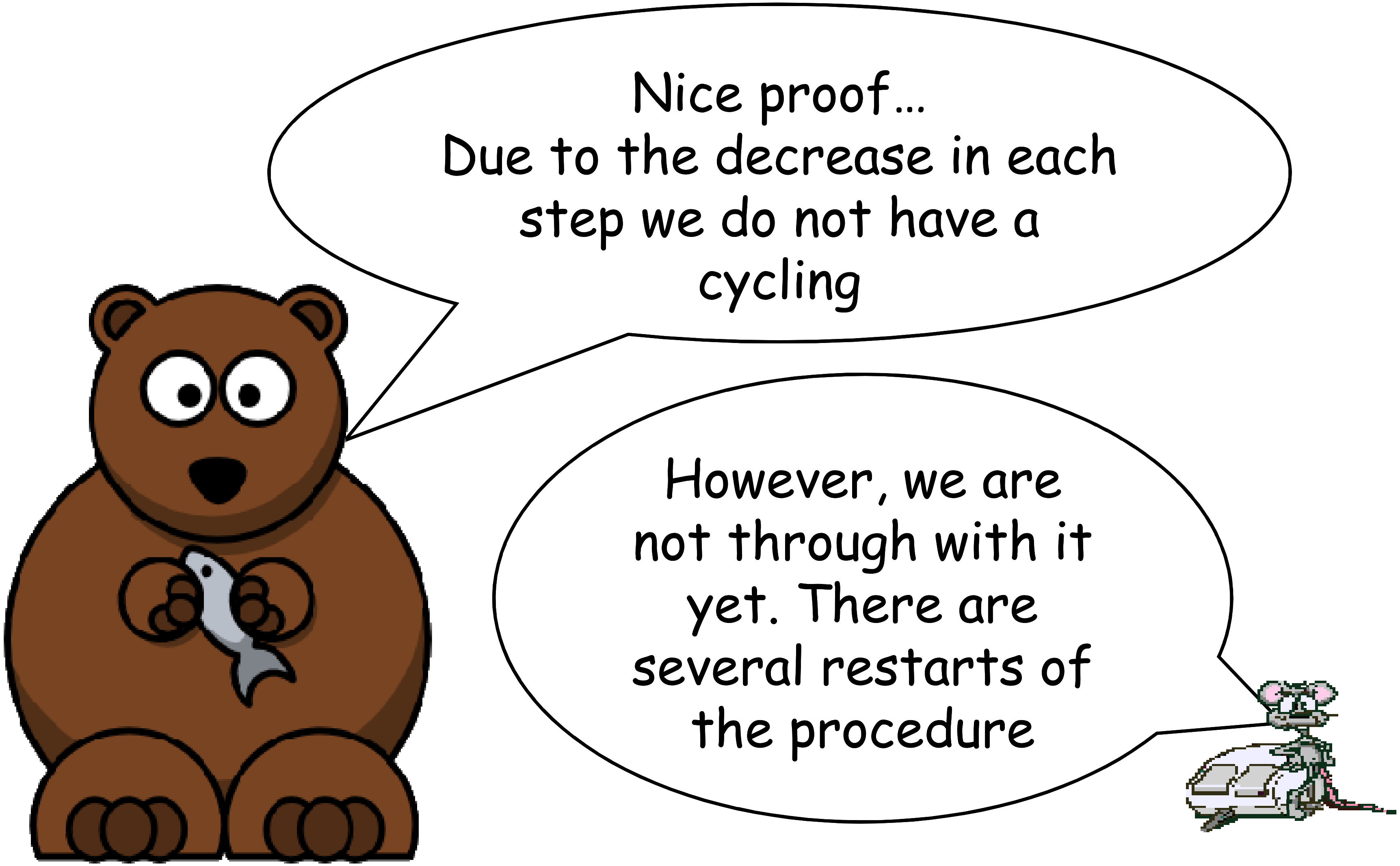
Clearly, if it holds that $\alpha_i^t = 0$ we have $\tilde{\alpha}_i^0 = \alpha_i^0$.

We consider the lowest numbered row i with $\alpha_i^t \neq 0$. Since α^t is lex-positive,

we conclude $\alpha_i^t > 0$ and due to $\dfrac{\alpha_i^t \cdot \alpha_{i_0}^0}{\alpha_{i_0}^t} > 0$, we conclude $\tilde{\alpha}_i^0 < \alpha_i^0$.

Hence, the column zero lex-decreases in each iteration of the dual simplex algorithm.

Finiteness of the algorithm



Proof of Theorem 9.2.3

- Clearly, between two applications of the dual simplex algorithm an additional row is added to the tableau
- This additional restriction reduces the set of feasible solutions
- Moreover, in each step of the dual simplex the column zero strictly lex-decreases

Let A_i^k be the ith column of the tableau matrix after the kth execution of the dual simplex algorithm. Due to the aforementioned attributes, we conclude that $A_0^1 >^L A_0^2 >^L A_0^3 >^L \ldots >^L A_0^l$

Proof of Theorem 9.2.3

We have assumed that the problem is bounded. Therefore, the first component $a_{0,0}$ converges towards some number $w_{0,0}$ with the following definition: $w_{0,0} = \lfloor w_{0,0} \rfloor + f_{0,0}$

After a finite number of iterations $a_{0,0}$ falls below $\lfloor w_{0,0} \rfloor + 1$, and for some k we can write

$$a_{0,0}^{k} = \lfloor w_{0,0} \rfloor + f_{0,0}^{k}, \text{ with } f_{0,0}^{k} < 1$$

Consequently, this row provides the next cut

$$-f_{0,0}^{k} = -\sum_{j \notin B} f_{0,j}^{k} \cdot x_j + s$$

We then apply the dual simplex and choose column p to enter the basis.

After this pivot we obtain: $a_{0,0}^{k+1} = a_{0,0}^{k} - \dfrac{a_{0,p}^{k}}{f_{0,p}^{k}} \cdot f_{0,0}^{k}$

Proof of Theorem 9.2.3

Now, at an optimal tableau of the dual simplex we have

$$(1)\quad a_{0,p}^{k} \geq 0$$

and therefore it is larger than its fractional part

$$(2)\quad a_{0,p}^{k} \geq f_{0,p}^{k}$$

Hence, it holds that:

$$(3)\quad a_{0,0}^{k+1} = a_{0,0}^{k} - \frac{a_{0,p}^{k}}{f_{0,p}^{k}} \cdot f_{0,0}^{k} \leq a_{0,0}^{k} - \frac{a_{0,p}^{k}}{a_{0,p}^{k}} \cdot f_{0,0}^{k} = a_{0,0}^{k} - f_{0,0}^{k} = \left\lfloor a_{0,0}^{k} \right\rfloor = \left\lfloor w_{0,0}^{k} \right\rfloor$$

Due to the convergence of the sequence $a_{0,0}^{l}$ to $w_{0,0}$, this shows that from this point on $a_{0,0}^{k} = \left\lfloor w_{0,0}^{k} \right\rfloor$ is an integer.

Proof of Theorem 9.2.3

The vectors A_0^l are lex-decreasing, and we have shown that after some point the first component becomes fixed at an integer. Consequently, the second component is monotonically non-increasing. It is lower bounded by zero.

The argument above can then be repeated for $a_{1,p}^l$.

However, we need to show that $a_{1,p}^k \geq 0$ so that the steps following step (2) go through. This follows because $a_{0,0}^k$ remains fixed, which implies that $a_{0,p}^k = 0$.

This implies $a_{1,p}^k \geq 0$ because $A_p^k >^L 0$.

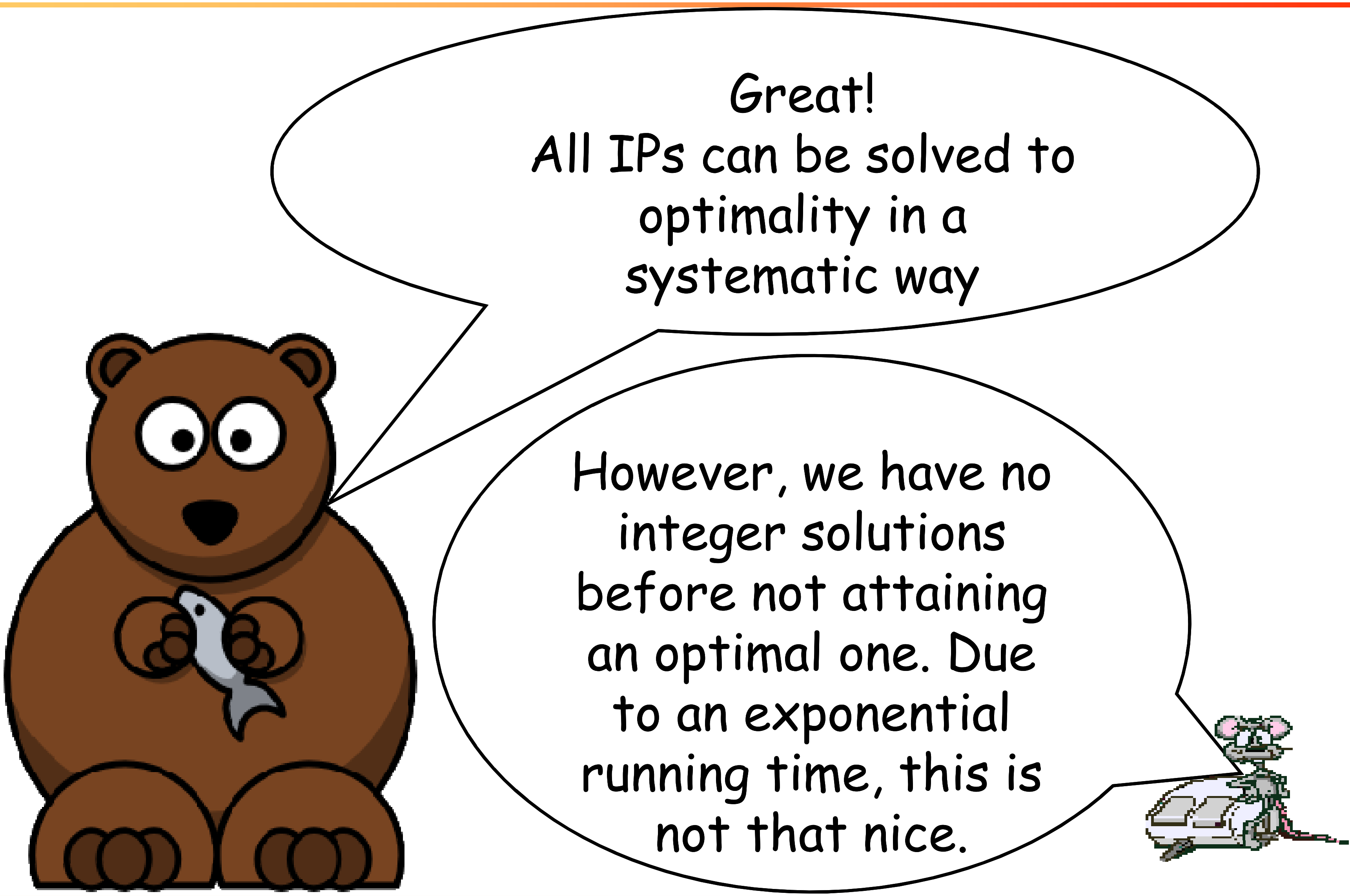
Hence, $a_{1,0}^l$ becomes integer after a finite number of steps.

We can continue in this way down column zero, showing that all components eventually reach integer values, at which point the algorithm terminates. The only other possible termination occurs when the dual simplex algorithm finds that the dual is unbounded, and hence that the original ILP is infeasible.

Proof of Theorem 9.2.3

- Moreover, an indefinite number of rows and columns is avoided by dropping a slack variable of a cut if it becomes fractional and is associated with a new Gomory cut (by entering the basis)
- Consequently, we have always at most n rows and at most n-m additional cuts
- Since it was shown that the first column is strictly lex-decreasing during the computation, the number of considered constellations is bounded by an exponential function
- Consequently, the procedure terminates after a finite number of steps

Optimally solving Integer Programs (IPs)



Example

$$\text{Maximize } -1\cdot x_2$$

$$\text{s.t. } 3\cdot x_1 + 2\cdot x_2 \le 6$$

$$-3\cdot x_1 + 2\cdot x_2 \le 0$$

$$x_1, x_2 \ge 0 \wedge x_1, x_2 \in \mathbb{N}$$

We obtain for the LP-relaxation of the IP:

$$\begin{array}{c|cccc} 0 & 0 & -1 & 0 & 0 \\ \hline 6 & 3 & 2 & 1 & 0 \\ 0 & -3 & [2] & 0 & 1 \end{array} \Rightarrow \begin{array}{c|cccc} 0 & -3/2 & 0 & 0 & 1/2 \\ \hline 6 & [6] & 0 & 1 & -1 \\ 0 & -3/2 & 1 & 0 & 1/2 \end{array} \Rightarrow \begin{array}{c|cccc} 3/2 & 0 & 0 & 1/4 & 1/4 \\ \hline 1 & 1 & 0 & 1/6 & -1/6 \\ 3/2 & 0 & 1 & 1/4 & 1/4 \end{array}$$

Example (lexicographic algorithm)

$$\begin{array}{c|cccc} \frac{3}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ \hline 1 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} \\ \frac{3}{2} & 0 & 1 & \frac{1}{4} & \frac{1}{4} \end{array} \Rightarrow \begin{array}{c|ccccc} \frac{3}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \hline 1 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{3}{2} & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{array} \leftarrow Cut$$

$$i_0 = 3 \wedge a_3^0 = -\frac{1}{2}$$

$$\frac{\alpha^3}{-a_3^3} = \frac{\begin{pmatrix} \frac{1}{4} & \frac{1}{6} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}^T}{\frac{1}{4}} = \begin{pmatrix} 1 & \frac{2}{3} & 1 & -1 \end{pmatrix}^T$$

$$\frac{\alpha^4}{-a_3^4} = \frac{\begin{pmatrix} \frac{1}{4} & -\frac{1}{6} & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}^T}{\frac{1}{4}} = \begin{pmatrix} 1 & -\frac{2}{3} & 1 & -1 \end{pmatrix}^T$$

$$\Rightarrow \frac{\alpha^3}{-a_3^3} - \frac{\alpha^4}{-a_3^4} = \begin{pmatrix} 1 & \frac{2}{3} & 1 & -1 \end{pmatrix}^T - \begin{pmatrix} 1 & -\frac{2}{3} & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 0 & \frac{4}{3} & 0 & 0 \end{pmatrix}^T > 0 \Rightarrow \frac{\alpha^3}{-a_3^3} >^L \frac{\alpha^4}{-a_3^4}$$

Thus, we resume with the fourth column

Example

$$
\begin{array}{c|cccccc}
\frac{3}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\
\hline
1 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\
\frac{3}{2} & 0 & 1 & \frac{1}{4} & \frac{1}{4} & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{4} & \left(-\frac{1}{4}\right) & 1
\end{array}
\Rightarrow
\begin{array}{c|ccccc}
1 & 0 & 0 & 0 & 0 & 1 \\
\hline
\frac{4}{3} & 1 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} \\
1 & 0 & 1 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 & 1 & -4
\end{array}
$$

- We obtain the optimal LP solution $x^T = (4/3,1,0,2,0)$
- Consequently, we add an additional restriction resulting from the second row

Example

$$\begin{array}{c|ccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ \hline 4/3 & 1 & 0 & 1/3 & 0 & -2/3 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 & -4 \end{array} \Rightarrow \begin{array}{c|cccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 4/3 & 1 & 0 & 1/3 & 0 & -2/3 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 1 & -4 & 0 \\ -1/3 & 0 & 0 & -1/3 & 0 & -1/3 & 1 \end{array} \leftarrow Cut$$

$$i_0 = 4 \wedge a_4^0 = -1/3$$

$$\frac{\alpha^3}{-a_4^3} = \frac{\begin{pmatrix}0 & 1/3 & 0 & 1 & -1/3\end{pmatrix}^T}{1/3} = \begin{pmatrix}0 & 1 & 0 & 3 & -1\end{pmatrix}^T$$

$$\frac{\alpha^5}{-a_4^5} = \frac{\begin{pmatrix}1 & -2/3 & 1 & -4 & -1/3\end{pmatrix}^T}{1/3} = \begin{pmatrix}3 & -2 & 3 & -12 & -1\end{pmatrix}^T$$

$$\Rightarrow \frac{\alpha^5}{-a_4^5} - \frac{\alpha^3}{-a_4^3} = \begin{pmatrix}3 & -2 & 3 & -12 & -1\end{pmatrix}^T - \begin{pmatrix}0 & 1 & 0 & 3 & -1\end{pmatrix}^T = \begin{pmatrix}3 & -3 & 3 & -15 & 0\end{pmatrix}^T \Rightarrow \frac{\alpha^5}{-a_4^5} \overset{L}{>} \frac{\alpha^3}{-a_4^3}$$

Thus, we resume with the third column

Example

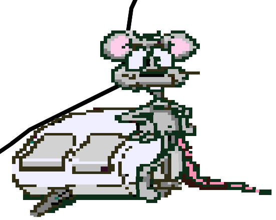
$$
\begin{array}{c|cccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
\frac{4}{3} & 1 & 0 & \frac{1}{3} & 0 & -\frac{2}{3} & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & 1 & -4 & 0 \\
-\frac{1}{3} & 0 & 0 & \left(-\frac{1}{3}\right) & 0 & -\frac{1}{3} & 1
\end{array}
\Rightarrow
\begin{array}{c|cccccc}
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & -1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & -5 & 3 \\
1 & 0 & 0 & 1 & 0 & 1 & -3
\end{array}
$$

The optimal solution to the original integer problem is $x^T = (1,1)$

Optimally solving Integer Programs (IPs)

What about systematically bounding variables that are not integer if we do not see adequate cuts?

Yes. You are right. This is considered next: Branch&Bound! We can even combine that Branch&Bound technique with the generation of cuts



9.3 Branch&Bound

- In what follows, we consider a second technique optimally solving general integer linear programs with a bounded solution space. Given an integer Linear Program denoted as M^0

$$(M^0)\ Min\ z(x) \quad s.t.\ x \in P^0$$

- We consider a **lower bound** LM^0 to M^0 that is obtained from a relaxation and has a larger solution space $LP^0 \supseteq P^0$. We solve the relaxation to optimality and obtain its optimal solution x^0

$$LM^0 = Min\ z(x) \quad s.t.\ x \in LP^0$$

For example, LM^0 is the optimal objective function value of the LP-relaxation to M^0

- If $x^0 \in P^0$, then the problem M^0 is optimally solved.
- Otherwise: **Branching** (see next slide)

Branching

We partition the solution space P^0 by some branching rule and yield k+1 subproblems $M^{00}\ldots M^{0k}$

$$P^0 = \bigcup_{i=1}^{k} P^{0i} \quad \wedge \quad \forall i, j = 0,\ldots,k : i \neq j : P^{0i} \cap P^{0j} = \emptyset$$

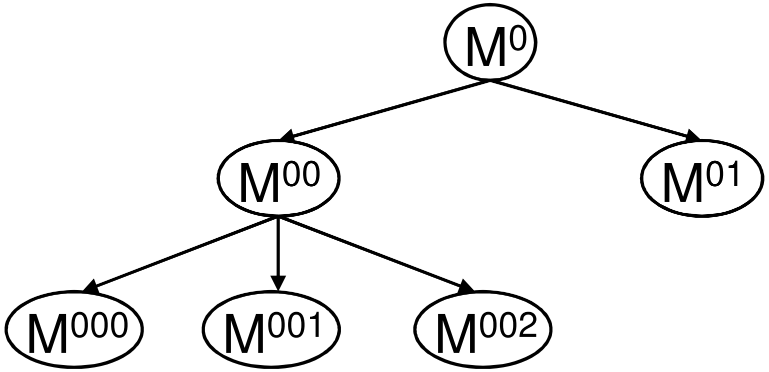
$$\left(M^{00}\right) Min\ z(x) \ \ s.t.\ x \in P^{01} \quad \ldots \quad \left(M^{0k}\right) Min\ z(x) \ \ s.t.\ x \in P^{0k}$$

For example, if P^0 is the LP-relaxation, we choose a variable x_j^0 that is **not** integer and yield two subproblems with

$$P^{00} = \left\{ x \geq 0 \middle| x \in P^0 \wedge x_j \geq \left\lceil x_j^0 \right\rceil \right\} \qquad P^{01} = \left\{ x \geq 0 \middle| x \in P^0 \wedge x_j \leq \left\lfloor x_j^0 \right\rfloor \right\}$$

Enumeration tree obtained from Branching

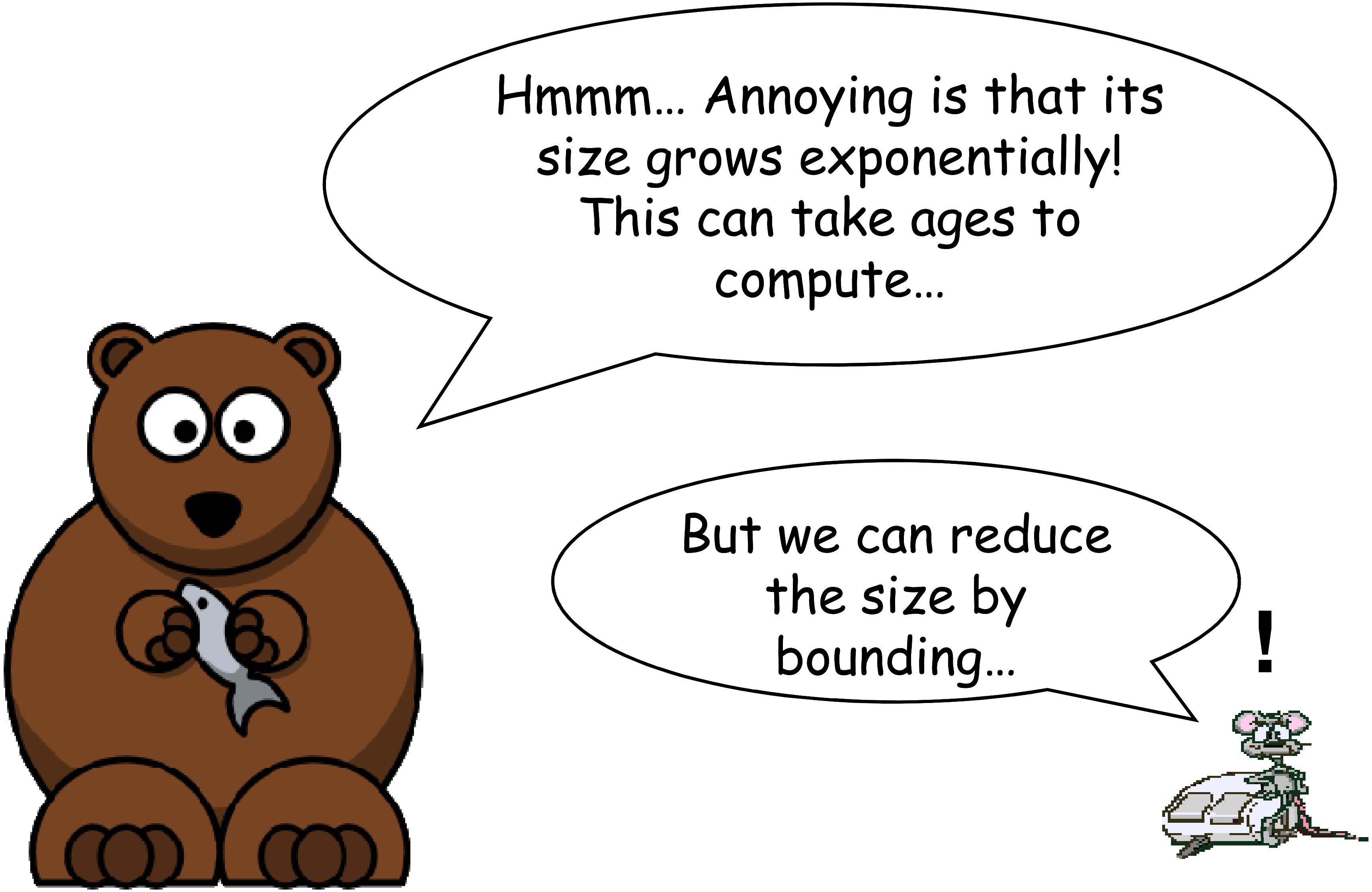
Applying the branching rule consecutively, we derive a solution tree



Some solutions to the subproblems may be integer.

We stop if the solution tree is explored entirely, and thus the best known integer solution is optimal to M^0.

Size of the enumeration tree



Bounding

There is always a **global upper bound** UM to the integer Linear Program M^0. Either UM=∞ or UM is derived from a feasible solution to M^0

We calculate a lower bound LM^{0i}, which is easy to calculate, for each subproblem M^{0i} , and LM^{0i} has a solution space $LP^{0i} \supseteq P^{0i}$ $\forall i=1,\ldots,k$.

A subproblem M^{0i} does not need to be considered anymore (i.e., it is pruned) if **one** of the following **pruning criterions** holds:

a) $LM^{0i} < UM$ and the optimal solution x^{0i} of LM^{0i} is feasible to M^0: We found an improved upper bound to M^0 , and we remember this solution UM:= LM^{0i}.

b) $LM^{0i} \geq UM$: The optimal solution to the subproblem M^{0i} , and all integer solutions derived from it cannot be better than the best known feasible solution with UM.

c) $LP^{0i} = \varnothing$: There exists no feasible solution to LM^{0i} and none to M^{0i}.

We stop if the solution tree is explored, and thus UM is optimal to M^0.

Example

$$(M^0) \quad \text{Minimize } -x_1 - 2\cdot x_2$$

$$\text{s.t.} \quad 2\cdot x_1 + 2\cdot x_2 \le 7$$

$$-2\cdot x_1 + 2\cdot x_2 \le 1$$

$$-2\cdot x_2 \le -1$$

$$x_1, x_2 \ge 0$$

$$x_1, x_2 \in \mathbb{Z}$$

We commence with UM=∞ and with the LP-relaxation LM^0

$$\begin{array}{c|ccccc} 0 & -1 & -2 & 0 & 0 & 0 \\ \hline 7 & 2 & 2 & 1 & 0 & 0 \\ 1 & -2 & 2 & 0 & 1 & 0 \\ -1 & 0 & -2 & 0 & 0 & 1 \end{array} \Rightarrow \ldots \Rightarrow \begin{array}{c|ccccc} 11\tfrac{1}{2} & 0 & 0 & \tfrac{3}{4} & \tfrac{1}{4} & 0 \\ \hline \tfrac{3}{2} & 1 & 0 & \tfrac{1}{4} & -\tfrac{1}{4} & 0 \\ 3 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 \\ 2 & 0 & 1 & \tfrac{1}{4} & \tfrac{1}{4} & 0 \end{array}$$

Consequences

- Obviously, -11/2 is a lower bound for the optimal solution value of M^0
- Since the solution is unfortunately not integer, we branch and conduct a case statement. Either $x_1 \leq 1$ or $x_1 \geq 2$
- Starting from the original set of feasible solutions

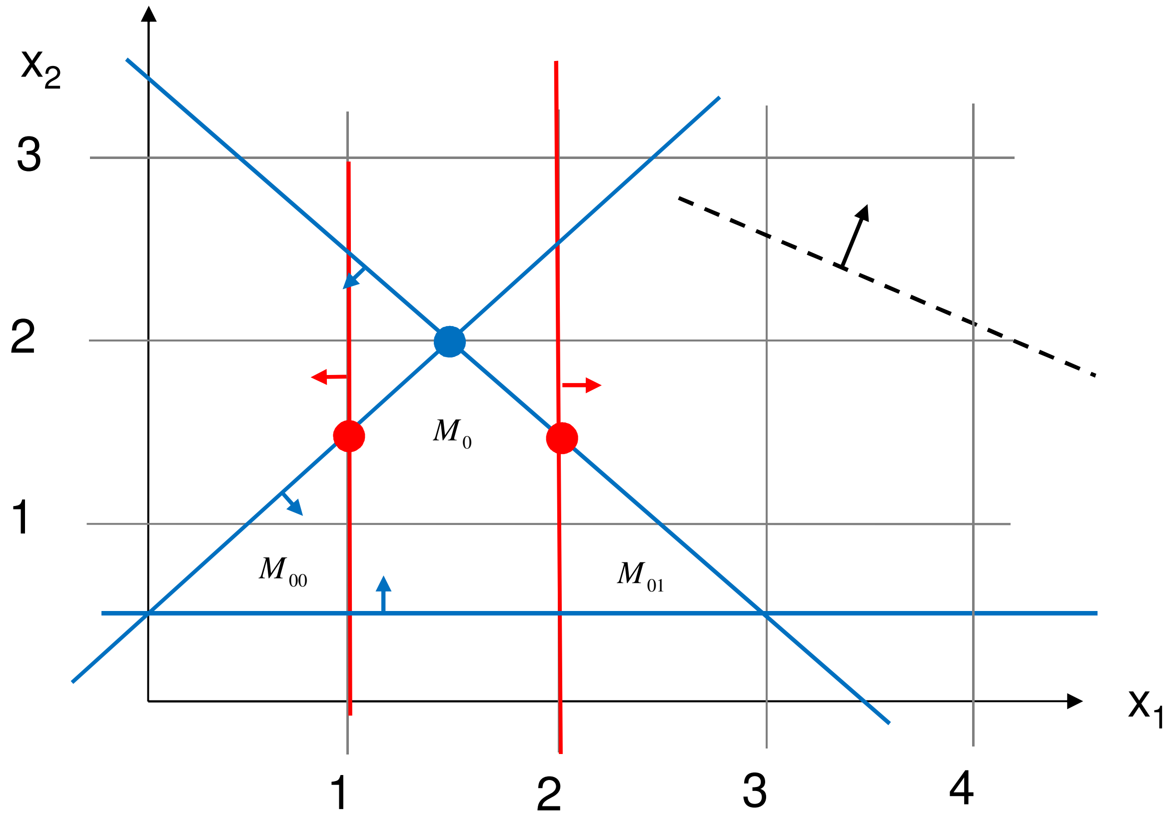
$$P^0 = \left\{ (x_1, x_2) \in IR^2_{\geq 0} \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \right\}$$

the simple branching step yields two subproblems

$$P^{00} = \left\{ (x_1, x_2) \in IR^2_{\geq 0} \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \leq 1 \right\} \wedge$$

$$P^{01} = \left\{ (x_1, x_2) \in IR^2_{\geq 0} \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \geq 2 \right\}$$

First Level Branches



Resulting problems

Consequently, we obtain the tableaus

M^{00}

$$\begin{array}{c|cccccc}
11\tfrac{1}{2} & 0 & 0 & \tfrac{3}{4} & \tfrac{1}{4} & 0 & 0 \\
\hline
\tfrac{3}{2} & 1 & 0 & \tfrac{1}{4} & -\tfrac{1}{4} & 0 & 0 \\
3 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 & 0 \\
2 & 0 & 1 & \tfrac{1}{4} & \tfrac{1}{2} & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}$$

M^{01}

$$\begin{array}{c|cccccc}
11\tfrac{1}{2} & 0 & 0 & \tfrac{3}{4} & \tfrac{1}{4} & 0 & 0 \\
\hline
\tfrac{3}{2} & 1 & 0 & \tfrac{1}{4} & -\tfrac{1}{4} & 0 & 0 \\
3 & 0 & 0 & \tfrac{1}{2} & \tfrac{1}{2} & 1 & 0 \\
2 & 0 & 1 & \tfrac{1}{4} & \tfrac{1}{2} & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & -1
\end{array}$$

Transformation of the tableaus

- In order to directly conduct the Dual Simplex, we need to transform the problem
- Specifically, we subtract the first row from the fourth one or vice versa
- Thus, we obtain

$$
\begin{array}{c|cccccc}
11/2 & 0 & 0 & 3/4 & 1/4 & 0 & 0 \\
\hline
3/2 & 1 & 0 & 1/4 & -1/4 & 0 & 0 \\
3 & 0 & 0 & 1/2 & 1/2 & 1 & 0 \\
2 & 0 & 1 & 1/4 & 1/2 & 0 & 0 \\
-1/2 & 0 & 0 & -1/4 & 1/4 & 0 & 1
\end{array}
\qquad
\begin{array}{c|cccccc}
11/2 & 0 & 0 & 3/4 & 1/4 & 0 & 0 \\
\hline
3/2 & 1 & 0 & 1/4 & -1/4 & 0 & 0 \\
3 & 0 & 0 & 1/2 & 1/2 & 1 & 0 \\
2 & 0 & 1 & 1/4 & 1/2 & 0 & 0 \\
-1/2 & 0 & 0 & 1/4 & -1/4 & 0 & 1
\end{array}
$$

Finally, it turns out…

$$
\begin{array}{c|cccccc}
11/2 & 0 & 0 & 3/4 & 1/4 & 0 & 0 \\
\hline
3/2 & 1 & 0 & 1/4 & -1/4 & 0 & 0 \\
3 & 0 & 0 & 1/2 & 1/2 & 1 & 0 \\
2 & 0 & 1 & 1/4 & 1/4 & 0 & 0 \\
\left[-1/2\right] & 0 & 0 & \left(-1/4\right) & 1/4 & 0 & 1
\end{array}
\qquad
\begin{array}{c|cccccc}
11/2 & 0 & 0 & 3/4 & 1/4 & 0 & 0 \\
\hline
3/2 & 1 & 0 & 1/4 & -1/4 & 0 & 0 \\
3 & 0 & 0 & 1/2 & 1/2 & 1 & 0 \\
2 & 0 & 1 & 1/4 & 1/4 & 0 & 0 \\
\left[-1/2\right] & 0 & 0 & 1/4 & \left(-1/4\right) & 0 & 1
\end{array}
$$

$$
\begin{array}{c|cccccc}
4 & 0 & 0 & 0 & 1 & 0 & 3 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 & 1 & 2 \\
3/2 & 0 & 1 & 0 & 1/2 & 0 & 1 \\
2 & 0 & 0 & 1 & -1 & 0 & -4
\end{array}
\qquad
\begin{array}{c|cccccc}
5 & 0 & 0 & 1 & 0 & 0 & 1 \\
\hline
2 & 1 & 0 & 0 & 0 & 0 & -1 \\
2 & 0 & 0 & 1 & 0 & 1 & 2 \\
3/2 & 0 & 1 & 1/2 & 0 & 0 & 1 \\
2 & 0 & 0 & -1 & 1 & 0 & -4
\end{array}
$$

Conclusions

- Unfortunately, both solutions are still not integer
- Thus, we have to resume with the next branching step
- This time, we obtain altogether four constellations

$$M^{000} = \left\{ \left(x_1, x_2 \right) \in IR_{\geq 0}^2 \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \leq 1 \wedge x_2 \leq 1 \right\} \wedge$$

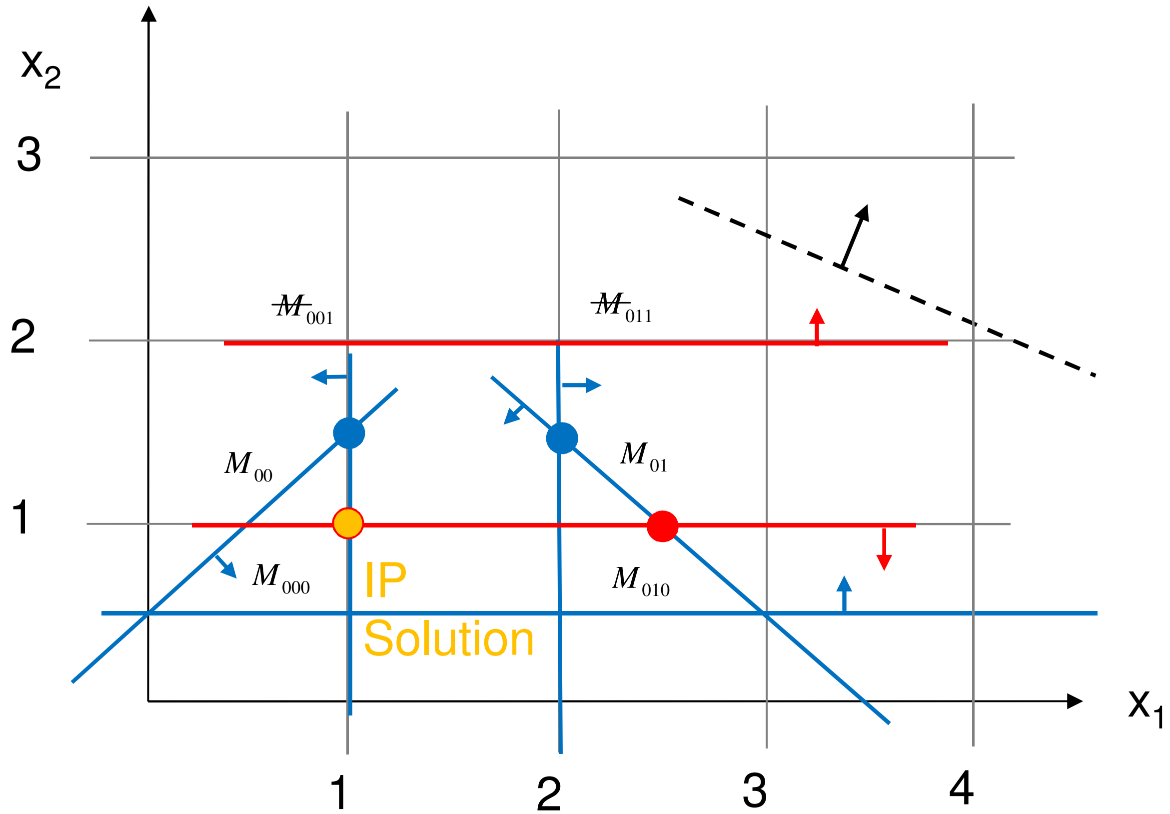
$$M^{001} = \left\{ \left(x_1, x_2 \right) \in IR_{\geq 0}^2 \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \leq 1 \wedge x_2 \geq 2 \right\} \wedge$$

$$M^{010} = \left\{ \left(x_1, x_2 \right) \in IR_{\geq 0}^2 \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \geq 2 \wedge x_2 \leq 1 \right\} \wedge$$

$$M^{011} = \left\{ \left(x_1, x_2 \right) \in IR_{\geq 0}^2 \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \geq 2 \wedge x_2 \geq 2 \right\}$$

- M^{001} and M^{011} are infeasible (case c)
- Thus, we resume with M^{000} and M^{010}

Second Level Branches



Resulting problems

M^{000}

$$
\begin{array}{c|ccccccc}
4 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\
\frac{3}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & -1 & 0 & -4 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

M^{010}

$$
\begin{array}{c|ccccccc}
5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\
\frac{3}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & -1 & 1 & 0 & -4 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

$$
\begin{array}{c|ccccccc}
4 & 0 & 0 & 0 & 1 & 0 & 3 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 \\
\frac{3}{2} & 0 & 1 & 0 & \frac{1}{2} & 0 & 1 & 0 \\
2 & 0 & 0 & 1 & -1 & 0 & -4 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & \left(-\frac{1}{2}\right) & 0 & -1 & 1
\end{array}
$$

$$
\begin{array}{c|ccccccc}
5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
2 & 0 & 0 & 1 & 0 & 1 & 2 & 0 \\
\frac{3}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\
2 & 0 & 0 & -1 & 1 & 0 & -4 & 0 \\
-\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & (-1) & 1
\end{array}
$$

Resulting problems

4	0	0	0	1	0	3	0
1	1	0	0	0	0	1	0
2	0	0	0	1	1	2	0
$3/2$	0	1	0	$1/2$	0	1	0
2	0	0	1	-1	0	-4	0
$\left[-1/2\right]$	0	0	0	$\left(-1/2\right)$	0	-1	1

5	0	0	1	0	0	1	0
2	1	0	0	0	0	-1	0
2	0	0	1	0	1	2	0
$3/2$	0	1	$1/2$	0	0	1	0
2	0	0	-1	1	0	-4	0
$\left[-1/2\right]$	0	0	$-1/2$	0	0	(-1)	1

3	0	0	0	0	0	1	2
1	1	0	0	0	0	1	0
1	0	0	0	0	1	0	2
1	0	1	0	0	0	0	1
3	0	0	1	0	0	-2	-2
1	0	0	0	1	0	2	-2

$9/2$	0	0	$1/2$	0	0	0	1
$5/2$	1	0	$1/2$	0	0	0	-1
1	0	0	0	0	1	0	2
1	0	1	0	0	0	0	1
4	0	0	1	1	0	0	-4
$1/2$	0	0	$1/2$	0	0	1	-1

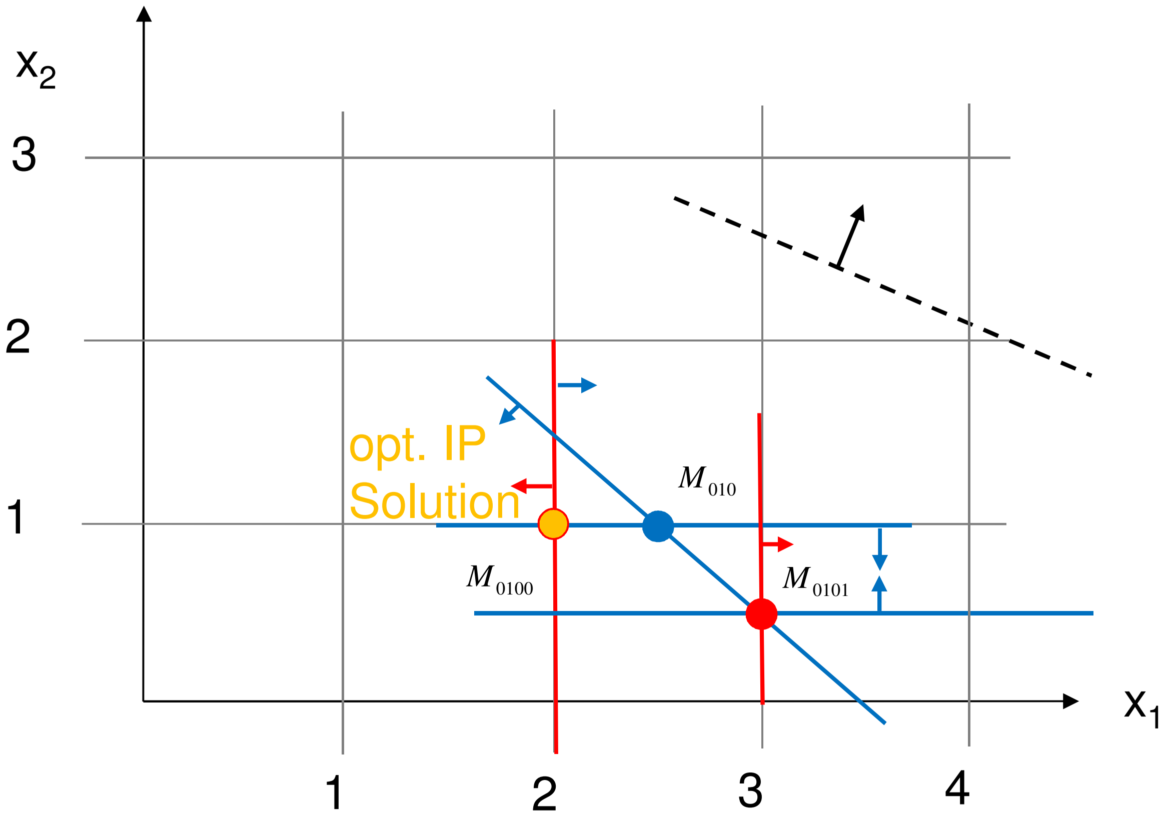
M^{000} and M^{010} – Results

- Obviously, the problems are optimally solved
- Thus, we obtain an integer solution with objective function value -3 from M^{000} and we set UM:=-3 (case a)
- Since the lower bound of the remaining problem M^{010} is -9/2, we have to resume with this problem
- Here, we obtain the new problems

$$M^{0100} = \left\{ \left(x_1, x_2\right) \in IR^2_{\geq 0} \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \geq 2 \wedge x_2 \leq 1 \wedge x_1 \leq 2 \right\} \wedge$$

$$M^{0101} = \left\{ \left(x_1, x_2\right) \in IR^2_{\geq 0} \mid 2 \cdot x_1 + 2 \cdot x_2 \leq 7 \wedge -2 \cdot x_1 + 2 \cdot x_2 \leq 1 \wedge -2 \cdot x_2 \leq -1 \wedge x_1 \geq 2 \wedge x_2 \leq 1 \wedge x_1 \geq 3 \right\}$$

Third Level Branches



M^{010} – Results

M^{0100}

$$
\begin{array}{c|cccccccc}
9/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1 & 0 \\
\hline
5/2 & 1 & 0 & 1/2 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 1 & -1 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

M^{0101}

$$
\begin{array}{c|cccccccc}
9/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1 & 0 \\
\hline
5/2 & 1 & 0 & 1/2 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 1 & -1 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{array}
$$

$$
\begin{array}{c|cccccccc}
9/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1 & 0 \\
\hline
5/2 & 1 & 0 & 1/2 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 1 & -1 & 0 \\
-1/2 & 0 & 0 & -1/2 & 0 & 0 & 0 & 1 & 1
\end{array}
$$

$$
\begin{array}{c|cccccccc}
9/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & 1 & 0 \\
\hline
5/2 & 1 & 0 & 1/2 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
1/2 & 0 & 0 & 1/2 & 0 & 0 & 1 & -1 & 0 \\
-1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 & -1 & 1
\end{array}
$$

And thus, we obtain

$$
\begin{array}{c|cccccccc}
\frac{9}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\
\hline
\frac{5}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & -1 & 0 \\
\left[-\frac{1}{2}\right] & 0 & 0 & \left(-\frac{1}{2}\right) & 0 & 0 & 0 & 1 & 1
\end{array}
\qquad
\begin{array}{c|cccccccc}
\frac{9}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\
\hline
\frac{5}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & -4 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 1 & -1 & 0 \\
\left[-\frac{1}{2}\right] & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & (-1) & 1
\end{array}
$$

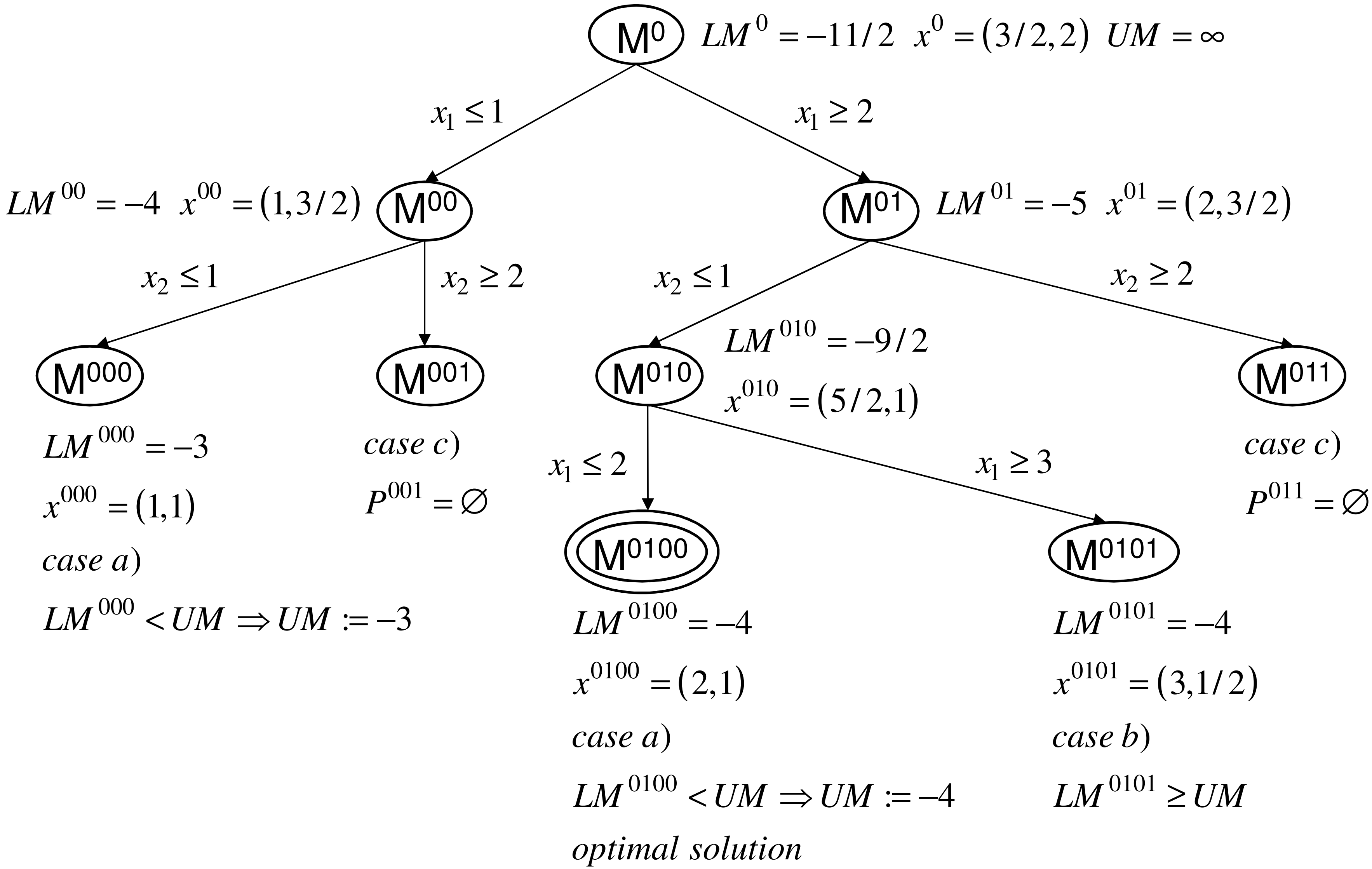
$$
\begin{array}{c|cccccccc}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
\hline
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2
\end{array}
\qquad
\begin{array}{c|cccccccc}
4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 \\
\frac{1}{2} & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \\
6 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -4 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & -1
\end{array}
$$

M^{0100} and M^{0101} – Results

- We obtained an improved second feasible solution $x^T=(2,1)$ from M^{0100} and UM:=-4 (case a)
- The other alternative constellation M^{0101} still does not provide any integer solution

  However, since the objective function value is -4, this is a lower bound for all integer solutions resulting from M^{0101} (case b)
- Thus, we explored the solution tree and stop our procedure. The optimal solution is $x^T=(2,1)$ with an objective function value of UM=-4

Example – Conducted exploration process



Branch&Bound Algorithm

1. Determine an upper bound UM either via a heuristic or set UM:=∞.
2. Solve a lower bound LM^i of M^i and obtain its optimal solution x^i .
3. If either $LM^i \geq UM$ (case b) or $P^i = \emptyset$ (case c) holds, then go to 7.
4. Otherwise (case b) or c) do not apply): If $LM^i < UM$ and x^i is feasible to M^0 (case a), then set $UM := LM^i$. Check for each remaining candidate problem M^k that is in the list whether it can be pruned by $LM^k \geq UM$ (case b). Remove all pruned problems M^k from the list. Go to 7.
5. Otherwise (case a) does not apply): LM^i is a candidate problem and is stored in a list.
6. Pick a candidate problem M^k from the list. Branch the problem M^k and derive a subproblem M^{ki}. If no subproblem is derived, then remove M^k from the list. Proceed with $M^i := M^{ki}$ and go to 2.
7. If there exists no candidate problem in the list, then terminate the algorithm. The optimal solution is the corresponding solution to UM.
8. Otherwise (there exist candidate problems in the list): Go to 6.