10 Matrix games

- In what, follows, we provide a brief introduction to Game Theory
- Specifically, we consider specific games that are definable as Linear Programs
- This will lead to specific games, in the following denoted as *Matrix Games*
- The matrix, the basic structure of the game, defines the payments resulting from the chosen policies of the players
- Player are, for instance, persons, companies, states (i.e., their governments)

10.1 Introducing examples

- In what follows, we introduce two possible applications that are representative for Matrix Games
- By means of these applications, we will derive optimal strategies

Typical applications are

- Well-known two-person game "paper-rock-scissors"
- Location planning
- As mentioned above, these games are completely defined by their respective matrices

10.1.1 Application: "rock-paper-scissors"

- In each move, two players may either select rock, paper, or scissors
- These selections are simultaneously executed and are completely independent of each other
- In order to prevent simple optimal strategies of the players, the following priority rules are applied

Priority rules:

- Rock beats scissors, but is defeated by paper
- Scissors beats paper, but is defeated by rock
- Paper beats rock, but is defeated by scissors

Thus, we obtain the results

P_2/P_1	P_1 selects rock	P_1 selects scissors	P_1 selects paper
P_2 selects rock	draw	P_2 wins	P_1 wins
P_2 selects scissors	P_1 wins	draw	P_2 wins
P_2 selects paper	P_2 wins	P_1 wins	draw

Matrix of the game

Thus, we obtain the following matrix that determines the results of the first player.

Specifically, we define:

1 = Player 1 wins; 0 = draw; -1 = Player 2 wins

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

Moves of the players

With the matrix of results $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$,

which is defined from the point of view of player 1, we define $x \in \{0,1\}^3$ as the choice of player 1, and $y \in \{0,1\}^3$ as the choice of player 2. Since each player has to provide a definite decision, we require:

$$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3 = 1 \wedge \sum_{i=1}^{3} y_i = y_1 + y_2 + y_3 = 1$$

The result of a move

Thus, $A \cdot x \in IR^3$ determines the possible results player 1 will obtain by his choice depending on the choice of player 2.

Alternatively, $\left(y^T \cdot A\right)^T \in IR^3$ determines the possible results player 1 will obtain by the choice of player 2 depending on the choice of player 1.

Thus, we can calculate the resulting payment of player 1 depending on first player's choice $(\text{i.e., } x)$ as well as on second player's choice $(\text{i.e., } y)$ by $y^T \cdot A \cdot x$.

10.1.2 Application: Bilateral monopole

- Two vendors, namely A and B, want to erect an additional store in a certain common sales area
- Depending on their choices, vendors A and B attain different profits
- Specifically, the area is separated in altogether four regions and again the vendors take their decisions independently
- Again, we consider the situation out of the position of player A
- Player B pursues a minimization of the profit attained by player A

The attainable profits of vendor A

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	44	72	64	64
B selects region 2	68	58	60	65
B selects region 3	64	68	72	75
B selects region 4	56	64	61	59

Possible strategies

- Vendor A may chose a max-min strategy
- I.e., A tries to maximize the profit that is minimally attainable, or, with other words, tries to optimize its profit in a worst case constellation
- Vendor B may chose a min-max strategy
- I.e., B tries to minimize the profit that is maximally reachable by A, or, with other words, tries to minimize the profit that A attains in its best case constellation

Thus, the max-min strategy provides for A

- A with max-min
  - Region 1: min{44,68,64,56}=44
  - Region 2: min{72,58,68,64}=58
  - Region 3: min{64,60,72,61}=60
  - Region 4: min{64,65,75,59}=59
- Thus, we obtain max{44,58,60,59}=60
- Consequently, applying the max-min strategy, A would take region 3 with the minimum profit of 60

The resulting profits of vendor A

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	44	72	64	64
B selects region 2	68	58	**60**	65
B selects region 3	64	68	72	75
B selects region 4	56	64	61	59

The min-max strategy provides for B

- B with min-max
  - Region 1: max{44,72,64,64}=72
  - Region 2: max{68,58,60,65}=68
  - Region 3: max{64,68,72,75}=75
  - Region 4: max{56,64,61,59}=64
- Thus, we obtain min{72,68,75,64}=64
- Consequently, applying the min-max policy, B would select region 4, which limits the maximum profit of A to 64

Consequence

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	44	72	64	64
B selects region 2	68	58	60	65
B selects region 3	64	68	72	75
B selects region 4	56	64	**61**	59

Observations

- Obviously, vendor A expects a minimum profit of 60, but eventually attains 61
- On the other hand, vendor B was willing to “accept” a profit of A of “even 64”, or better spoken, has already calculated it
- However, what can we learn from this example?
- What does the obtained result provide about the quality of the max-min strategy applied by vendor A?
- Are there any provable optimal strategies?

10.2 Basic definitions

10.2.1 Definition

*A **game** is a private, economic, social, or political competition. Components of such a game are*

1. ***Players**. These may be persons, companies, states, nature, or coincidences*
2. ***Moves**. These are selected by players according to predetermined rules of the game out of a finite set of alternatives*
3. ***Strategies**. They either determine the selection of activities entirely (pure strategy) or provide probabilities by that an activity is selected (mixed strategies). For the latter ones repetition is necessary*
4. ***Payments**. They define resulting yields or losses of the opponents under specific moves*

Two-person zero-sum games

10.2.2 Definition

*A game with two opponents, denoted as players, where one person wins what the other loses. It is denoted as a **two-person zero-sum game** or simpler just **matrix game**.*

*Moreover, **a two-person zero-sum game is denoted as symmetric** if both players select their moves out of an identical reservoir of activities and if their roles are exchangeable.*

Assumption

10.2.3 Definition

In what follows, the payment matrix is always defined out of the view of player 1. In this connection, the ith column gives the profits according to the choice of player 2 if player 1 selects the ith alternative. Analogously, the jth row gives the profits according to the choice of player 1 if player 2 selects the jth alternative.

Scope of strategies

10.2.4 Definition:

Let $S^{(n)} = \{x \in IR^n \mid x \geq 0 \wedge 1^T \cdot x = 1\}$ be the set of strategies, i.e.,

x_i and π_i determine the probability of chosing the ith move for

$x \in S^{(n)}$ resp. $\pi \in S^{(m)}$.

Furthermore, pure strategies are characterized by the fact that probabilities are always 1.

Consequences

10.2.5 Lemma:

1. Assuming the players act independently, we know

$$P\begin{pmatrix}\text{Player 1 choses alternative no. } i \wedge \\ \text{Player 2 choses alternative no. } j\end{pmatrix} = x_i \cdot \pi_j$$

2. The expected value of the payments is determined by:

$$E(x,\pi) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \cdot \pi_j \cdot a_{i,j} = \sum_{i=1}^{n}\left(\sum_{j=1}^{m} \pi_j \cdot a_{i,j}\right) \cdot x_i = \pi^T \cdot A \cdot x$$

Further definition

10.2.6 Definition

In what follows, we define

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1,\ldots,m\} \mid j = 1,\ldots,n\}$$
$$a^0 = \min\{\max\{a_{i,j} \mid j = 1,\ldots,n\} \mid i = 1,\ldots,m\}$$

and

$$M_0 = \max\{\min\{E(x,\pi) \mid \pi \in S^{(m)}\} \mid x \in S^{(n)}\}$$
$$M^0 = \min\{\max\{E(x,\pi) \mid x \in S^{(n)}\} \mid \pi \in S^{(m)}\}$$

Fair games

10.2.7 Definition:

A game is denoted as fair if $M_0 = 0$.

A corresponding strategy x_0 is denoted as optimal if $M_0 = \min\{E(x_0,\pi) \mid \pi \in S^{(m)}\}$.

In addition to this, π^0 is denoted as optimal if $M^0 = \max\{E(x,\pi^0) \mid x \in S^{(n)}\}$.

Observations

- An optimal strategy of player 1 attains at least the maximum of all expected minimum profits if player 2 plays optimally, i.e., if player 2 implements a worst case scenario for player 1
- Other way round, player 2 plays optimally if he is able to at least minimize the expected maximum profit of player 1 if this player acts optimally
- However, if the competing player does not apply such a strategy, there may be better results
- Note that the optimal strategy is generated for the case in that the opponent plays optimally

Applied to the two examples

$$A = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix}$$

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1,...,m\} \mid j = 1,...,n\} = \max\{44,58,60,59\} = 60$$

$$\wedge\, a^0 = \min\{\max\{a_{i,j} \mid j = 1,...,n\} \mid i = 1,...,m\} = \min\{72,68,75,64\} = 64$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1,...,m\} \mid j = 1,...,n\} = \max\{-1,-1,-1\} = -1$$

$$\wedge\, a^0 = \min\{\max\{a_{i,j} \mid j = 1,...,n\} \mid i = 1,...,m\} = \min\{1,1,1\} = 1$$

Consequence

Let $x \in S^{(n)}$ be a strategy of player 1 and let $\pi \in S^{(m)}$ be a strategy of player 2 . Then, with $A \in IR^{(m \times n)}$, it holds:

$$E(x,\pi) = \pi^T \cdot A \cdot x = \pi^T \cdot (A \cdot x) = \left(\sum_{j=1}^{m} \pi_j \cdot e^j \right)^T \cdot A \cdot x = \sum_{j=1}^{m} \pi_j \cdot \left(e^j\right)^T \cdot A \cdot x$$

We know that $\sum_{j=1}^{m} \pi_j = 1$. Thus, it holds:

$$E(x,\pi) = \sum_{j=1}^{m} \pi_j \cdot \left(e^j\right)^T \cdot A \cdot x \geq \min\left\{ \left(e^j\right)^T \cdot A \cdot x \mid j \in \{1,...,m\} \right\}$$

Thus, there exists a pure strategy of player 2 that is optimal for a given mixed strategy of player 1.

Analogous consequence

Let $x \in S^{(n)}$ be a strategy of player 1 and let $\pi \in S^{(m)}$ be a strategy of player 2 . Then, with $A \in IR^{(m \times n)}$, it holds:

$$E(x,\pi) = \pi^T \cdot A \cdot x = \pi^T \cdot (A \cdot x) = \pi^T \cdot \left(\sum_{i=1}^{n} a_{j,i} \cdot x_i \right)_{1 \leq j \leq m} = \pi^T \cdot \sum_{i=1}^{n} x_i \cdot e^i \cdot A$$

We know that $\sum_{i=1}^{n} x_i = 1$. Thus, it holds:

$$E(x,\pi) = \pi^T \cdot \sum_{i=1}^{n} x_i \cdot e^i \cdot A \leq \max\left\{ \pi^T \cdot A \cdot e^i \mid i \in \{1,...,n\} \right\}$$

Thus, there exists a pure strategy of player 1 that is optimal for a given mixed strategy of player 2.

Conclusion

10.2.8 Lemma:

It holds:

$$a_0 \leq M_0 \leq M^0 \leq a^0$$

Proof of Lemma 10.2.8

$$
\begin{aligned}
a_0 &= \max\left\{\min\left\{a_{i,j} | i = 1,\ldots,m\right\} | j = 1,\ldots,n\right\} \\
&= \max_{i \in \{1,\ldots,n\}} \min_{j \in \{1,\ldots,m\}} \left(\left(e^j\right)^T \cdot A\right) \cdot e^i \\
&= \max_{e_i} \min_{e_j} \left(\left(e^j\right)^T \cdot A\right) \cdot e^i \\
&\leq \max_x \min_\pi \left(\pi^T \cdot A\right) \cdot x = M_0
\end{aligned}
$$

$$
\begin{aligned}
M^0 &= \min_\pi \max_x \pi^T \cdot (A \cdot x) \leq \min_{e_j} \max_{e_i} \left(\left(e^j\right)^T \cdot A\right) \cdot e^i \\
&= \min_{j \in \{1,\ldots,m\}} \max_{i \in \{1,\ldots,n\}} \left(\left(e^j\right)^T \cdot A\right) \cdot e^i \\
&= \min\left\{\max\left\{a_{i,j} | j = 1,\ldots,n\right\} | i = 1,\ldots,m\right\} = a^0
\end{aligned}
$$

Proof of Lemma 10.2.8

Let x_0 and π_0 be optimal strategies. Then, we can conclude:

$$
\begin{aligned}
M_0 &= \max_x \left(\min_\pi \left(\pi^T \cdot A\right) \cdot x\right) = \min_\pi \left(\pi^T \cdot A\right) \cdot x_0 \\
&\leq {\pi_0}^T \cdot A \cdot x_0 \leq \max_x {\pi_0}^T \cdot A \cdot x = M^0
\end{aligned}
$$

$$\Rightarrow M_0 \leq M^0$$

Altogether, we obtain: $a_0 \leq M_0 \leq M^0 \leq a^0$

10.3 Games and Linear Programming

- In what follows, we provide methods that generate optimal strategies for two-person games
- These methods are based on the principles of Linear Programming
- Consequently, at first, we provide an LP-problem definition

Preliminary work I

10.3.1 Lemma

Assume $x \in S^{(n)}$ and $M \in IR$:

Then, it holds: $A \cdot x \geq M \cdot \underbrace{(1,\ldots,1)}_{m \text{ times}} \Leftrightarrow \min_{\pi} \pi^{T} \cdot A \cdot x \geq M$

Proof of Lemma 10.3.1

$$A \cdot x \geq M \cdot \underbrace{(1,\ldots,1)}_{m \text{ times}}$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^{T} \cdot A \cdot x \geq \pi^{T} \cdot M \cdot (1,\ldots,1), \text{ with } \sum_{j=1}^{m} \pi_{j} = 1$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^{T} \cdot A \cdot x \geq M \cdot \sum_{j=1}^{m} \pi_{j}$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^{T} \cdot A \cdot x \geq M \Leftrightarrow \min_{\pi} \pi^{T} \cdot A \cdot x \geq M$$

Preliminary work II

10.3.2 Lemma:

Assume $x \in S^{(n)}$ and $M \in IR$:

Then, it holds: $\pi^{T} \cdot A \leq M \cdot \underbrace{(1,\ldots,1)}_{n \text{ times}} \Leftrightarrow \max_{x} \pi^{T} \cdot A \cdot x \leq M$

Proof of Lemma 10.3.2

$$\pi^T \cdot A \le M \cdot \underbrace{(1,\dots,1)}_{n\text{ times}}$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \le M \cdot (1,\dots,1) \cdot x, \text{with } \sum_{i=1}^{n} x_i = 1$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \le M \cdot \sum_{i=1}^{n} x_i$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \le M \Leftrightarrow \max_x \pi^T \cdot A \cdot x \le M$$

Preliminary work III

Assume that x_0 is an optimal strategy for player 1. Then, we know that it holds: $\min_\pi \pi^T \cdot A \cdot x_0 = M_0$

By making use of Lemma 10.3.1, we conclude that it holds:

$A \cdot x_0 \ge M_0 \cdot 1_{(m)}$

Additionally, if it holds $M_0 > 0,$ we define $x_1 = \frac{x_0}{M_0}$ and obtain

$A \cdot x_1 = A \cdot \frac{x_0}{M_0} \ge 1_{(m)}.$ Since $x_0 \ge 0 \Rightarrow x_1 \ge 0.$

Preliminary work IV

Assume that π_0 is an optimal strategy for player 2. Then, we know $\max_x (\pi_0)^T \cdot A \cdot x = M^0$

By making use of Lemma 10.3.2, we conclude that it holds:

$(\pi_0)^T \cdot A \le M^0 \cdot 1_{(n)}$

Additionally, if it holds $M^0 > 0,$ we define $\pi_1 = \frac{\pi_0}{M^0}$ and obtain

$(\pi_1)^T \cdot A = \left(\frac{\pi_0}{M^0}\right)^T \cdot A \le 1_{(n)}.$ Since $\pi_0 \ge 0 \Rightarrow \pi_1 \ge 0.$

The Linear Program

We introduce the following Linear Program (P)

Minimize $1^T \cdot x$, s.t. $A \cdot x \geq 1 \wedge x \geq 0$

as the LP that corresponds to the game matrix A

Observations

10.3.3 Lemma:

1. Let x be feasible for (P) with $M = \frac{1}{1^T \cdot x}$, then $M > 0$ and for $\tilde{x} = M \cdot x$ it holds: $\tilde{x} \in S^{(n)} \wedge \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M$.
2. Other way round, if $\tilde{x} \in S^{(n)} \wedge M = \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} > 0$, then it holds that x is feasible for (P), with $x = \frac{\tilde{x}}{M}$, and $1^T \cdot x = \frac{1}{M}$.

Proof of Lemma 10.3.3

1. Let x be feasible for (P) with $M = \frac{1}{1^T \cdot x} \Rightarrow x \geq 0 \wedge A \cdot x \geq 1$
$\Rightarrow 1^T \cdot x > 0 \Rightarrow M = \frac{1}{1^T \cdot x} > 0$. Let $\tilde{x} = M \cdot x \Rightarrow A \cdot \tilde{x} = A \cdot M \cdot x$
$= M \cdot A \cdot x \geq M \cdot 1$.
Thus, we can apply Lemma 10.3.1 $\Rightarrow \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M$.
Additionally, we calculate $1^T \cdot \tilde{x} = 1^T \cdot M \cdot x = 1^T \cdot \frac{1}{1^T \cdot x} \cdot x = 1$
and obviously $\tilde{x} \geq 0$. Thus, $\tilde{x} \in S^{(n)}$.

Proof of Lemma 10.3.3

2. Let $\tilde{x} \in S^{(n)} \wedge \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M > 0 \Rightarrow \tilde{x} \geq 0 \wedge 1^T \cdot \tilde{x} = 1$

Since Lemma 10.3.1, it additionally holds $A \cdot \tilde{x} \geq M \cdot (1,\ldots,1)$

$\Rightarrow$ We define $x = \dfrac{\tilde{x}}{M} \Rightarrow A \cdot x = \dfrac{A \cdot \tilde{x}}{M} \geq \dfrac{M}{M} = 1 \Rightarrow A \cdot x \geq 1$

Since $M > 0$, it holds that $x = \dfrac{\tilde{x}}{M} \geq 0$.

Consequently, x is feasible for (P), with $1^T \cdot x = 1^T \cdot \dfrac{\tilde{x}}{M} = \dfrac{1}{M}$.

and for optimal solutions of the LP…

10.3.4 Lemma

Let x_0 be an optimal solution of LP. Then it holds:

$$1^T \cdot x_0 = \frac{1}{M_0}$$

Proof of Lemma 10.3.4

At first, we show that $\dfrac{1}{M_0}$ is a lower bound for the objective function $1^T \cdot x$ if the problem is solvable. Let x be some feasible solution for the LP. Furthermore, let $\tilde{x} = \dfrac{x}{M(x)}$, with $M(x) = 1^T \cdot x$. Thus, we conclude $A \cdot \tilde{x} = \dfrac{A \cdot x}{M(x)} \geq \dfrac{1}{M(x)}$

$\Rightarrow \min_\pi \pi^T \cdot A \cdot \tilde{x} \geq \dfrac{1}{M(x)}$. It holds: $M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x$

$\Rightarrow \dfrac{1}{M(x)} = \dfrac{1}{1^T \cdot x} \leq M_0 \Leftrightarrow 1^T \cdot x \geq \dfrac{1}{M_0}$.

Proof of Lemma 10.3.4

Now, other way round, let x_0 be an optimal solution to LP (P) as defined above.

Consider now $M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x = \min_\pi \pi^T \cdot A \cdot \tilde{x}$, with $\tilde{x} \in S^{(n)}$. Thus, by making use of Lemma 10.3.3(2), we know that $M_0 = \frac{1}{1^T \cdot \tilde{x}}$ and $x_1 = \tilde{x} \cdot \frac{1}{M_0}$ is a feasible solution to LP. Since x_0 is an optimal solution to LP, we obtain $1^T \cdot x_0 \le 1^T \cdot x_1$

$$= \underbrace{1^T \cdot \tilde{x}}_{=1,\text{ since } \tilde{x} \in S^{(n)}} \cdot \frac{1}{M_0} = \frac{1}{M_0}.$$

Consequently, we get: $1^T \cdot x_0 \le \frac{1}{M_0} \wedge 1^T \cdot x_0 \ge \frac{1}{M_0} \Rightarrow 1^T \cdot x_0 = \frac{1}{M_0}$.

Consequence

Obviously, if x_0 is an optimal solution to the LP, we know that $\tilde{x} = \frac{x_0}{1^T \cdot x_0} = x_0 \cdot M$ is a feasible strategy and we consider

$$A \cdot \tilde{x} = A \cdot \frac{x_0}{1^T \cdot x_0} = \frac{A \cdot x_0}{1^T \cdot x_0} \ge \frac{(1,...,1)}{1^T \cdot x_0} = (M,...,M).$$

Since x is minimally chosen, we get a maximal M and therefore $(M,...,M)$ is maximized. Consequently, we just maximize $\min_\pi \pi^T \cdot A \cdot x$. Unfortunately, if A is defined that way that $M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x \le 0$, the problem is not solvable.

Main Cognition

10.3.5 Theorem:

Assuming $M_0 > 0$. Then, x_0 is an optimal solution to the LP if and only if $\tilde{x}_0 = x_0 \cdot M_0 = \frac{x_0}{1^T \cdot x_0}$ is an optimal strategy for player 1.

Proof of Theorem 10.3.5

$\Rightarrow$

Let x_0 be an optimal solution to LP (P). Then, $\tilde{x}_0 = x_0 \cdot M_0 \in S^{(n)}$, with $M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x \geq \min_\pi \pi^T \cdot A \cdot \tilde{x}_0$. Other way round, we get:

$$\min_\pi \pi^T \cdot A \cdot \tilde{x}_0 = \min_\pi \pi^T \cdot A \cdot x_0 \cdot M_0$$

$$= \min_\pi \pi^T \cdot \underbrace{A \cdot x_0}_{\geq 1,\text{ since } x_0 \text{ is feasible for LP}} \cdot \frac{1}{1^T \cdot x_0}$$

$$\geq \min_\pi \pi^T \cdot \frac{(1,\ldots,1)^T}{1^T \cdot x_0} = \underbrace{\pi^T \cdot (1,\ldots,1)^T}_{=1,\text{ since } \pi \in S^{(m)}} \cdot \frac{1}{1^T \cdot x_0} = \frac{1}{1^T \cdot x_0} = M_0$$

Proof of Theorem 10.3.5

$\Rightarrow$

Consequently, we obtain

$$\min_\pi \pi^T \cdot A \cdot \tilde{x}_0 \leq M_0 \wedge \min_\pi \pi^T \cdot A \cdot \tilde{x}_0 \geq M_0$$

$$\Rightarrow \min_\pi \pi^T \cdot A \cdot \tilde{x}_0 = M_0$$

$\Rightarrow \tilde{x}_0 \in S^{(n)}$ is optimal!

Proof of Theorem 10.3.5

$\Leftarrow$

Let $\tilde{x}_0 \in S^{(n)}$ be an optimal strategy. Then $\min_\pi \pi^T \cdot A \cdot \tilde{x}_0 = M_0$. Thus, we make use of Lemma 10.3.3, and obtain

$A \cdot \tilde{x}_0 \geq M_0$. Consider $x_0 = \dfrac{\tilde{x}_0}{M_0}$.

It holds: $A \cdot x_0 = A \cdot \dfrac{\tilde{x}_0}{M_0} \geq \dfrac{M_0}{M_0} = 1$.

Thus, x_0 is feasible for LP. We calculate

$$1^T \cdot x_0 = 1^T \cdot \frac{\tilde{x}_0}{M_0} = \frac{1^T \cdot \tilde{x}_0}{M_0} = \frac{1}{M_0}.$$

Proof of Theorem 10.3.5

We know that $\frac{1}{M_0}$ is a lower bound for $1^T \cdot x$ if x is a feasible solution for LP (P).

Consequently, we have shown that x_0 is an optimal solution.

This completes the proof.

The dual program

- The LP introduced above has the following dual

Maximize $1^T \cdot \pi$,

s.t.

$\pi^T \cdot A \leq 1^T \wedge \pi \geq 0$

$\Rightarrow$ Obviously, $\pi = 0$ is a feasible solution to (D)

Further consequences

10.3.6 Corollary

The following propositions are equivalent

1. *(P) has a feasible solution*
2. *(P) has an optimal solution*
3. *(D) has an optimal solution*
4. *$M_0>0$*

Proof of Corollary 10.3.6

$1 \Leftrightarrow 4:$

Let x be a feasible solution to (LP). Then, we have

$A \cdot x \geq 1$ and thus $M = \min_\pi \pi^T \cdot A \cdot x \geq 1 \Rightarrow$

$M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x \geq M > 0.$

Other way round, if $M_0 = \max_x \min_\pi \pi^T \cdot A \cdot x > 0.$

Thus, it exists $\tilde{x}_0 : M_0 = \min_\pi \pi^T \cdot A \cdot \tilde{x}_0 \Rightarrow A \cdot \tilde{x}_0 \geq M_0.$

We introduce: $x_0 = \tilde{x}_0 \cdot \frac{1}{M_0} \geq 0$

$\Rightarrow A \cdot x_0 = A \cdot \tilde{x}_0 \cdot \frac{1}{M_0} \geq M_0 \cdot \frac{1}{M_0} = 1 \Rightarrow x_0$ is feasible for (LP).

Proof of Corollary 10.3.6

$1 \Leftrightarrow 2 \Leftrightarrow 3:$

Obviously, (D) is always solvable, e.g., by making use of $\pi = 0$, we have at least one feasible solution.

Thus, through Section 2.2, there remain two cases. Either (D) is unrestricted and, therefore, (LP) not solvable or (D) and (LP) have optimal solutions.

This completes the proof.

What to do if it holds that $M_0 \leq 0$?

- Obviously, if $M_0>0$, we have provided an instrument that generates optimal strategies
- But if $M_0 \leq 0$, nothing is won since LP (P) is obviously not solvable
- However, what can we do in such kind of situation?
- Obviously, it is matrix A that incorporates this problem. Thus, the question to be posed is how we can modify this matrix in order to ensure that $M_0>0$

Adding a constant to *A*

Let us add a constant C to all matrix entries. We consider the result of a game, i.e.,

$$\pi^T \cdot A \cdot x = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\pi_i \cdot \left(a_{i,j}+C\right)\right)\cdot x_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{m}\pi_i \cdot a_{i,j} + \left(\pi_i \cdot C\right)\right)\cdot x_j$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} x_j \cdot \pi_i \cdot a_{i,j} + C \cdot \underbrace{\sum_{j=1}^{n}\sum_{i=1}^{m} x_j \cdot \pi_i}_{=1} = \sum_{j=1}^{n}\sum_{i=1}^{m} x_j \cdot \pi_i \cdot a_{i,j} + C$$

$\Rightarrow$

We obtain modified proceeds, but optimality is kept unchanged.

Consequences

- By adding a constant, we may be able to obtain a matrix A_{mod} that fulfills $M_0>0$
- The game that corresponds to the modified matrix A_{mod} has identical optimal strategies
- Consequently, we only have to retransform the resulting profits at the end of the calculation process

What to add?

- Fortunately, we know that M_0 is just raised by the value/constant C that is added to A
- The problem is that M_0 is unknown beforehand
- Otherwise, we just would take $-M_0+\varepsilon$, with $\varepsilon>0$
- Consequently, we may take $-a_0+\varepsilon$ ($\varepsilon>0$), which is a lower bound of M_0

Further bounds to add

2. Possibility:

Sufficient conditions are: $A \geq 0 \wedge a^0 = \min_i \max_j a_{i,j} > 0$

Consider the vector $1^T \cdot \frac{1}{a^0} \Rightarrow A \cdot \left(1^T \cdot \frac{1}{a^0}\right) = \frac{1}{a^0} \cdot \left(A \cdot 1^T\right) \geq \frac{1}{a^0} \cdot a^0 \cdot (1,\ldots,1)$

$= (1,\ldots,1) \Rightarrow 1^T \cdot \frac{1}{a^0}$ is feasible for (LP)!

$\Rightarrow$ We have to add $\max\{-\min a_{i,j}, -a^0 + \varepsilon\}$, with $\varepsilon > 0$

Further bounds to add

3. Possibility:

Sufficient condition is: $a_1 = \min_i \sum_{j=1}^{n} a_{i,j} > 0$

Consider the vector $1^T \cdot \frac{1}{a_1} \Rightarrow A \cdot \left(1^T \cdot \frac{1}{a_1}\right) = \frac{1}{a_1} \cdot \left(A \cdot 1^T\right) \geq \frac{1}{a_1} \cdot a_1 \cdot (1,\ldots,1)$

$= (1,\ldots,1) \Rightarrow 1^T \cdot \frac{1}{a_1}$ is feasible for (LP)!

$\Rightarrow$ We have to add $-\min_i \left(\frac{1}{n} \cdot \left(\sum_{j=1}^{n} a_{i,j}\right)\right) + \varepsilon$, with $\varepsilon > 0$

Summary

We always add:

$$\boxed{\min\left\{-a_0 + \varepsilon, -\min a_{i,j}, -a^0 + \varepsilon, -\min_i \left(\frac{1}{n} \cdot \left(\sum_{j=1}^{n} a_{i,j}\right)\right) + \varepsilon\right\}}$$

Note that if it holds $M_0 >> 0$, the value above becomes negative and matrix A is reduced

10.3.7 Example

- We now come back to our two introducing examples 10.1.1 and 10.1.2
- We start with example 10.1.1
- This was the simple game “rock, scissors, and paper”

Example

$$A=\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\Rightarrow$$

1. $a_0=\max_i \min_j a_{i,j}=\max\{-1,-1,-1\}=-1\Rightarrow C=1+\varepsilon$

2. $\min_{i,j} a_{i,j}=-1$ and $a^0=\min_i \max_j a_{i,j}=\min\{1,1,1\}=1$

$\Rightarrow C=\max\{-\min_{i,j} a_{i,j},-a^0+\varepsilon\}=\max\{1,-1+\varepsilon\}=1$

3. $\min\left\{\frac{1}{3}\cdot 0,\frac{1}{3}\cdot 0,\frac{1}{3}\cdot 0\right\}=0\Rightarrow C=\varepsilon$

We take *C=1*

Thus, we obtain the following matrix

$$A=\begin{pmatrix}0 & -1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 0\end{pmatrix}\Rightarrow \tilde{A}=\begin{pmatrix}1 & 0 & 2\\ 2 & 1 & 0\\ 0 & 2 & 1\end{pmatrix}\Rightarrow \tilde{A}^T=\begin{pmatrix}1 & 2 & 0\\ 0 & 1 & 2\\ 2 & 0 & 1\end{pmatrix}$$

Maximize $1^T\cdot\pi \Leftrightarrow$ Minimize $-1^T\cdot\pi$, w.r.t., $\tilde{A}^T\cdot\pi\le 1\wedge\pi\ge 0$

We solve the dual problem by introducing slackness variables

Calculation…

$$\begin{array}{c|cccccc} 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 \end{array}$$

Calculation…

$$\begin{array}{c|cccccc} 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ \hline 1 & 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & -4 & 1 & -2 & 0 & 1 \end{array}$$

Calculation…

$$\begin{array}{c|cccccc} 0 & 0 & 0 & -3 & 1 & -1 & 0 \\ \hline -1 & 1 & 0 & -4 & 1 & -2 & 0 \\ 1 & 0 & 1 & 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 9 & -2 & 4 & 1 \end{array}$$

Calculation…

$$\begin{array}{c|cccccc}
0 & 0 & 0 & -3 & 1 & -1 & 0 \\ \hline
-1 & 1 & 0 & -4 & 1 & -2 & 0 \\
1 & 0 & 1 & 2 & 0 & 1 & 0 \\
1/3 & 0 & 0 & 1 & -2/9 & 4/9 & 1/9
\end{array}$$

Calculation…

$$\begin{array}{c|cccccc}
0 & 0 & 0 & -3 & 1 & -1 & 0 \\ \hline
1/3 & 1 & 0 & 0 & 1/9 & -2/9 & 4/9 \\
1/3 & 0 & 1 & 0 & 4/9 & 1/9 & -2/9 \\
1/3 & 0 & 0 & 1 & -2/9 & 4/9 & 1/9
\end{array}$$

Calculation…

$$\begin{array}{c|cccccc}
1 & 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ \hline
1/3 & 1 & 0 & 0 & 1/9 & -2/9 & 4/9 \\
1/3 & 0 & 1 & 0 & 4/9 & 1/9 & -2/9 \\
1/3 & 0 & 0 & 1 & -2/9 & 4/9 & 1/9
\end{array}$$

Result

First row : $c^T - c_B^T \cdot A_B^{-1} \cdot E \Rightarrow x^T = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \pi^T$

$M_0 = 1 - C = 1 - 1 = 0.$

One can analogously show…

10.3.8 Observation:

Assuming $M^0 > 0$. Then, π^0 is an optimal solution to the dual of (LP) if and only if $\tilde{\pi}^0 = \pi^0 \cdot M^0 = \frac{\pi^0}{1^T \cdot \pi^0}$ is an optimal strategy for player 2.

Fundamental Theorem of matrix games

10.3.9 Theorem:

1. There are always optimal strategies and for each pair of optimal strategies x_0 and π^0 it holds: $M_0 = \pi^{0T} \cdot A \cdot x_0 = M^0$.
2. Optimal strategies x_0 and π^0 always fulfill $\min_i e^{iT} \cdot A \cdot x_0 = M_0 = M^0 = \max_j \pi^{0T} \cdot A \cdot e^j$.

Proof of Theorem 10.3.9

ad 1.

By adding the value C, we obtain an optimally solvable LP, whose optimal solution always corresponds to an optimal strategy. Thus, we conclude the general solvability of matrix games.

We assume we have a pair of optimal strategies x_0 and π^0. Then, we know that $\frac{1}{M_0}$ and $\frac{1}{M^0}$ are objective function values of the primal and dual program, respectively. Thus, we conclude $M_0 = M^0$.

Additionally, we know:

$M_0 = \min_\pi \pi^T \cdot A \cdot x_0 \le \pi^{0T} \cdot A \cdot x_0 \le \max_x \pi^{0T} \cdot A \cdot x = M^0$

$\Rightarrow M_0 = \min_\pi \pi^T \cdot A \cdot x_0 = \pi^{0T} \cdot A \cdot x_0 = \max_x \pi^{0T} \cdot A \cdot x = M^0$

Proof of Theorem 10.3.9

ad 2.

Let x_0 and π^0 be optimal strategies. Then, we know $M_0 = \min_\pi \pi^T \cdot A \cdot x_0 = \max_x \pi_0^T \cdot A \cdot x = M^0$. In addition, we have $\min_\pi \pi^T \cdot A \cdot x_0 \le \min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_0$.

Obviously, we have $\forall \pi \in S^{(m)} : \pi = \sum_{i=1}^{m} \pi_i \cdot e^i$. Thus, we obtain:

$\min_\pi \pi^T \cdot A \cdot x_0 = \pi^{0T} \cdot A \cdot x_0 = \left(\sum_{i=1}^{m} \pi_i^0 \cdot e^i \right)^T \cdot A \cdot x_0 = \sum_{i=1}^{m} \pi_i^0 \cdot \left(e^{iT} \cdot A \right) \cdot x_0$

$\ge \min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_0 \Rightarrow \min_\pi \pi^T \cdot A \cdot x_0 = \min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_0$

Analogously, one can show $\max_x \pi_0^T \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi_0^T \cdot A \cdot e^j$.

Proof of Theorem 10.3.9

Hence, we can conclude:

$\min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_0 = \min_\pi \pi^T \cdot A \cdot x_0 = M_0 = M^0 = \max_x \pi^{0T} \cdot A \cdot x$

$= \max_{e^j \in S^{(n)}} \pi^{0T} \cdot A \cdot e^j$

Other way round, if $\min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_1 = \max_{e^j \in S^{(n)}} \pi^{1T} \cdot A \cdot e^j$ for a pair of strategies x_1 and π^1, we obtain

$M_0 = \min_\pi \max_x \pi^T \cdot A \cdot x \le \max_x \pi^{1T} \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi^{1T} \cdot A \cdot e^j$

$= \min_{e^i \in S^{(m)}} e^{iT} \cdot A \cdot x_1 \le \min_\pi \pi^T \cdot A \cdot x_1 \le \max_x \min_\pi \pi^T \cdot A \cdot x = M^0$

Proof of Theorem 10.3.9

$\Rightarrow$ Since $M_0 = M^0 \Rightarrow$

$$M_0 = \min_\pi \max_x \pi^T \cdot A \cdot x = \max_x \pi^{1^T} \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi^{1^T} \cdot A \cdot e^j$$

$$= \min_{e^i \in S^{(m)}} e^{i^T} \cdot A \cdot x_1 = \min_\pi \pi^T \cdot A \cdot x_1 = \max_x \min_\pi \pi^T \cdot A \cdot x = M^0$$

10.3.10 Example

Example 9.1.2:

$$A = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix} \Rightarrow$$

1. $a_0 = \max_i \min_j a_{i,j} = \max\{44, 58, 60, 59\} = 60 \Rightarrow C = -60 + \varepsilon$
2. $\min_{i,j} a_{i,j} = 44$ and $a^0 = \min_i \max_j a_{i,j} = \min\{72, 68, 75, 64\} = 64$

$\Rightarrow C = \max\{-\min_{i,j} a_{i,j}, -a^0 + \varepsilon\} = \max\{-44, -64 + \varepsilon\} = -44$

3. $\min\left\{-\frac{1}{4} \cdot 244, -\frac{1}{4} \cdot 251, -\frac{1}{4} \cdot 279, -\frac{1}{4} \cdot 240\right\} = -60 \Rightarrow C = -60 + \varepsilon$

We take $C = -59$

The resulting program

$$A - 59 = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix} - 59 = \begin{pmatrix} -15 & 13 & 5 & 5 \\ 9 & -1 & 1 & 6 \\ 5 & 9 & 13 & 16 \\ -3 & 5 & 2 & 0 \end{pmatrix} = \tilde{A}$$

$$\tilde{A}^T = \begin{pmatrix} -15 & 9 & 5 & -3 \\ 13 & -1 & 9 & 5 \\ 5 & 1 & 13 & 2 \\ 5 & 6 & 16 & 0 \end{pmatrix}$$

Optimal solution

By using $\tilde{A}^T = \begin{pmatrix} -15 & 9 & 5 & -3 \\ 13 & -1 & 9 & 5 \\ 5 & 1 & 13 & 2 \\ 5 & 6 & 16 & 0 \end{pmatrix}$,

we obtain the optimal solution

$$\tilde{x} = \left(0, \frac{1}{5}, 0, \frac{1}{5}\right)^T \wedge \tilde{\pi} = \left(0, \frac{1}{6}, 0, \frac{7}{30}\right)^T$$

Optimal solution – objective function value

$$\Rightarrow \tilde{\pi}^T \cdot \tilde{A} \cdot x = \left(0, \frac{1}{6}, 0, \frac{7}{30}\right) \cdot \begin{pmatrix} -15 & 13 & 5 & 5 \\ 9 & -1 & 1 & 6 \\ 5 & 9 & 13 & 16 \\ -3 & 5 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix}$$

$$= \left(\tfrac{9}{6} - \tfrac{21}{30} \quad -\tfrac{1}{6} + \tfrac{35}{30} \quad \tfrac{1}{6} + \tfrac{14}{30} \quad 1\right) \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix}$$

$$= \left(\tfrac{4}{5} \quad 1 \quad \tfrac{19}{30} \quad 1\right) \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} = \frac{2}{5}$$

Results

$$\tilde{M}_0 = \frac{1}{2/5} = \frac{5}{2} = \frac{1}{(1 \quad 1 \quad 1 \quad 1)^T \cdot \tilde{x}} = \frac{1}{\frac{1}{5} + \frac{1}{5}} \wedge$$

$$\tilde{M}^0 = \frac{1}{2/5} = \frac{5}{2} = \frac{1}{(1 \quad 1 \quad 1 \quad 1)^T \cdot \tilde{\pi}} = \frac{1}{\frac{1}{6} + \frac{7}{30}} = \frac{1}{\frac{12}{30}} = \frac{1}{\frac{2}{5}}$$

Transformation

With $\tilde{M}_0 = \frac{5}{2}$ we get

$$\tilde{x}_0 = \tilde{x} \cdot \frac{5}{2} = \begin{pmatrix} 0 & \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix}^T \cdot \frac{5}{2} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}^T \wedge$$

$$\tilde{\pi}^0 = \tilde{\pi} \cdot \frac{5}{2} = \begin{pmatrix} 0 & \frac{1}{6} & 0 & \frac{7}{30} \end{pmatrix}^T \cdot \frac{5}{2} = \begin{pmatrix} 0 & \frac{5}{12} & 0 & \frac{7}{12} \end{pmatrix}^T$$

$$\Rightarrow M_0 = M^0 = \frac{5}{2} + 59 = 61.5$$

$$\Rightarrow a_0 = 60 < M_0 = 61.5 = M^0 < a^0 = 64$$