

## 10 Matrix games

- In what, follows, we provide a brief introduction to Game Theory
- Specifically, we consider specific games that are definable as Linear Programs
- This will lead to specific games, in the following denoted as *Matrix Games*
- The matrix, the basic structure of the game, defines the payments resulting from the chosen policies of the players
- Player are, for instance, persons, companies, states (i.e., their governments)

## 10.1 Introducing examples

- In what follows, we introduce two possible applications that are representative for Matrix Games
  - By means of these applications, we will derive optimal strategies
- Typical applications are
- Well-known two-person game “paper-rock-scissors”
  - Location planning
  - As mentioned above, these games are completely defined by their respective matrices

### 10.1.1 Application: “rock-paper-scissors”

- In each move, two players may either select rock, paper, or scissors
- These selections are simultaneously executed and are completely independent of each other
- In order to prevent simple optimal strategies of the players, the following priority rules are applied

Priority rules:

- Rock beats scissors, but is defeated by paper
- Scissors beats paper, but is defeated by rock
- Paper beats rock, but is defeated by scissors

### Thus, we obtain the results

$P_2/P_1$	$P_1$ selects rock	$P_1$ selects scissors	$P_1$ selects paper
$P_2$ selects rock	draw	$P_2$ wins	$P_1$ wins
$P_2$ selects scissors	$P_1$ wins	draw	$P_2$ wins
$P_2$ selects paper	$P_2$ wins	$P_1$ wins	draw

## Matrix of the game

Thus, we obtain the following matrix that determines the results of the first player.

Specifically, we define :

1 = Player 1 wins; 0 = draw; -1 = Player 2 wins

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

## Moves of the players

With the matrix of results  $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ ,

which is defined from the point of view of player 1, we define  $x \in \{0,1\}^3$  as the choice of player 1, and  $y \in \{0,1\}^3$  as the choice of player 2. Since each player has to provide a definite decision, we require:

$$\sum_{i=1}^3 x_i = x_1 + x_2 + x_3 = 1 \wedge \sum_{i=1}^3 y_i = y_1 + y_2 + y_3 = 1$$

## The result of a move

Thus,  $A \cdot x \in \mathbb{R}^3$  determines the possible results player 1 will obtain by his choice depending on the choice of player 2.

Alternatively,  $(y^T \cdot A)^T \in \mathbb{R}^3$  determines the possible results player 1 will obtain by the choice of player 2 depending on the choice of player 1.

Thus, we can calculate the resulting payment of player 1 depending on first player's choice (i.e.,  $x$ ) as well as on second player's choice (i.e.,  $y$ ) by  $y^T \cdot A \cdot x$ .

## 10.1.2 Application: Bilateral monopoly

- Two vendors, namely A and B, want to erect an additional store in a certain common sales area
- Depending on their choices, vendors A and B attain different profits
- Specifically, the area is separated in altogether four regions and again the vendors take their decisions independently
- Again, we consider the situation out of the position of player A
- Player B pursues a minimization of the profit attained by player A

### The attainable profits of vendor A

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	44	72	64	64
B selects region 2	68	58	60	65
B selects region 3	64	68	72	75
B selects region 4	56	64	61	59

### Possible strategies

- Vendor A may chose a max-min strategy
- I.e., A tries to maximize the profit that is minimally attainable, or, with other words, tries to optimize its profit in a worst case constellation
- Vendor B may chose a min-max strategy
- I.e., B tries to minimize the profit that is maximally reachable by A, or, with other words, tries to minimize the profit that A attains in its best case constellation

### Thus, the max-min strategy provides for A

- A with max-min
  - Region 1:  $\min\{44, 68, 64, 56\} = 44$
  - Region 2:  $\min\{72, 58, 68, 64\} = 58$
  - Region 3:  $\min\{64, 60, 72, 61\} = 60$
  - Region 4:  $\min\{64, 65, 75, 59\} = 59$
- Thus, we obtain  $\max\{44, 58, 60, 59\} = 60$
- Consequently, applying the max-min strategy, A would take region 3 with the minimum profit of 60

### The resulting profits of vendor A

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	<u>44</u>	72	64	64
B selects region 2	68	<u>58</u>	<u>60</u>	65
B selects region 3	64	68	72	75
B selects region 4	56	64	61	<u>59</u>

## The min-max strategy provides for B

- B with min-max
  - Region 1:  $\max\{44, 72, 64, 64\}=72$
  - Region 2:  $\max\{68, 58, 60, 65\}=68$
  - Region 3:  $\max\{64, 68, 72, 75\}=75$
  - Region 4:  $\max\{56, 64, 61, 59\}=64$
- Thus, we obtain  $\min\{72, 68, 75, 64\}=64$
- Consequently, applying the min-max policy, B would select region 4, which limits the maximum profit of A to 64

## Consequence

B/A	A selects region 1	A selects region 2	A selects region 3	A selects region 4
B selects region 1	44	72	64	64
B selects region 2	68	58	60	65
B selects region 3	64	68	72	75
B selects region 4	56	64	<u>61</u>	59

## Observations

- Obviously, vendor A expects a minimum profit of 60, but eventually attains 61
- On the other hand, vendor B was willing to “accept” a profit of A of “even 64”, or better spoken, has already calculated it
- However, what can we learn from this example?
- What does the obtained result provide about the quality of the max-min strategy applied by vendor A?
- Are there any provable optimal strategies?

## 10.2 Basic definitions

### 10.2.1 Definition

A **game** is a private, economic, social, or political competition.

Components of such a game are

1. **Players.** These may be persons, companies, states, nature, or coincidences
2. **Moves.** These are selected by players according to predetermined rules of the game out of a finite set of alternatives
3. **Strategies.** They either determine the selection of activities entirely (pure strategy) or provide probabilities by that an activity is selected (mixed strategies). For the latter ones repetition is necessary
4. **Payments.** They define resulting yields or losses of the opponents under specific moves

## Two-person zero-sum games

### 10.2.2 Definition

A game with two opponents, denoted as players, where one person wins what the other loses. It is denoted as a **two-person zero-sum game** or simpler just **matrix game**.

Moreover, a **two-person zero-sum game** is denoted as **symmetric** if both players select their moves out of an identical reservoir of activities and if their roles are exchangeable.

## Assumption

### 10.2.3 Definition

In what follows, the payment matrix is always defined out of the view of player 1. In this connection, the  $i$ th column gives the profits according to the choice of player 2 if player 1 selects the  $i$ th alternative. Analogously, the  $j$ th row gives the profits according to the choice of player 1 if player 2 selects the  $j$ th alternative.

## Scope of strategies

### 10.2.4 Definition:

Let  $S^{(n)} = \{x \in IR^n \mid x \geq 0 \wedge 1^T \cdot x = 1\}$  be the set of strategies, i.e.,

$x_i$  and  $\pi_i$  determine the probability of choosing the  $i$ th move for

$x \in S^{(n)}$  resp.  $\pi \in S^{(m)}$ .

Furthermore, pure strategies are characterized by the fact that probabilities are always 1.

## Consequences

### 10.2.5 Lemma:

1. Assuming the players act independently, we know

$$P\left(\begin{array}{l} \text{Player 1 choses alternative no. } i \wedge \\ \text{Player 2 choses alternative no. } j \end{array}\right) = x_i \cdot \pi_j$$

2. The expected value of the payments is determined by:

$$E(x, \pi) = \sum_{i=1}^n \sum_{j=1}^m x_i \cdot \pi_j \cdot a_{ij} = \sum_{i=1}^n \left( \sum_{j=1}^m \pi_j \cdot a_{ij} \right) \cdot x_i = \pi^T \cdot A \cdot x$$

## Further definition

### 10.2.6 Definition

In what follows, we define

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1, \dots, m\} \mid j = 1, \dots, n\}$$

$$a^0 = \min\{\max\{a_{i,j} \mid j = 1, \dots, n\} \mid i = 1, \dots, m\}$$

and

$$M_0 = \max\{\min\{E(x, \pi) \mid \pi \in S^{(m)}\} \mid x \in S^{(n)}\}$$

$$M^0 = \min\{\max\{E(x, \pi) \mid x \in S^{(n)}\} \mid \pi \in S^{(m)}\}$$

## Fair games

### 10.2.7 Definition:

A game is denoted as fair if  $M_0 = 0$ .

A corresponding strategy  $x_0$  is denoted as optimal if

$$M_0 = \min\{E(x_0, \pi) \mid \pi \in S^{(m)}\}$$

In addition to this,  $\pi^0$  is denoted as optimal if

$$M^0 = \max\{E(x, \pi^0) \mid x \in S^{(n)}\}$$

## Observations

- An optimal strategy of player 1 attains at least the maximum of all expected minimum profits if player 2 plays optimally, i.e., if player 2 implements a worst case scenario for player 1
- Other way round, player 2 plays optimally if he is able to at least minimize the expected maximum profit of player 1 if this player acts optimally
- However, if the competing player does not apply such a strategy, there may be better results
- Note that the optimal strategy is generated for the case in that the opponent plays optimally

## Applied to the two examples

$$A = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix}$$

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1, \dots, m\} \mid j = 1, \dots, n\} = \max\{44, 58, 60, 59\} = 60$$

$$\wedge a^0 = \min\{\max\{a_{i,j} \mid j = 1, \dots, n\} \mid i = 1, \dots, m\} = \min\{72, 68, 75, 64\} = 64$$

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$a_0 = \max\{\min\{a_{i,j} \mid i = 1, \dots, m\} \mid j = 1, \dots, n\} = \max\{-1, -1, -1\} = -1$$

$$\wedge a^0 = \min\{\max\{a_{i,j} \mid j = 1, \dots, n\} \mid i = 1, \dots, m\} = \min\{1, 1, 1\} = 1$$

## Consequence

Let  $x \in S^{(n)}$  be a strategy of player 1 and let  $\pi \in S^{(m)}$  be a strategy of player 2. Then, with  $A \in IR^{(m \times n)}$ , it holds:

$$E(x, \pi) = \pi^T \cdot A \cdot x = \pi^T \cdot (A \cdot x) = \left( \sum_{j=1}^m \pi_j \cdot e^j \right)^T \cdot A \cdot x = \sum_{j=1}^m \pi_j \cdot (e^j)^T \cdot A \cdot x$$

We know that  $\sum_{j=1}^m \pi_j = 1$ . Thus, it holds:

$$E(x, \pi) = \sum_{j=1}^m \pi_j \cdot (e^j)^T \cdot A \cdot x \geq \min \left\{ (e^j)^T \cdot A \cdot x \mid j \in \{1, \dots, m\} \right\}$$

Thus, there exists a pure strategy of player 2 that is optimal for a given mixed strategy of player 1.

## Analogous consequence

Let  $x \in S^{(n)}$  be a strategy of player 1 and let  $\pi \in S^{(m)}$  be a strategy of player 2. Then, with  $A \in IR^{(m \times n)}$ , it holds:

$$E(x, \pi) = \pi^T \cdot A \cdot x = \pi^T \cdot (A \cdot x) = \pi^T \cdot \left( \sum_{i=1}^n a_{j,i} \cdot x_i \right)_{1 \leq j \leq m} = \pi^T \cdot \sum_{i=1}^n x_i \cdot e^i \cdot A$$

We know that  $\sum_{i=1}^n x_i = 1$ . Thus, it holds:

$$E(x, \pi) = \pi^T \cdot \sum_{i=1}^n x_i \cdot e^i \cdot A \leq \max \left\{ \pi^T \cdot A \cdot e^i \mid i \in \{1, \dots, n\} \right\}$$

Thus, there exists a pure strategy of player 1 that is optimal for a given mixed strategy of player 2.

## Conclusion

### 10.2.8 Lemma:

It holds:

$$a_0 \leq M_0 \leq M^0 \leq a^0$$

## Proof of Lemma 10.2.8

$$\begin{aligned} a_0 &= \max \left\{ \min \{ a_{i,j} \mid i = 1, \dots, m \} \mid j = 1, \dots, n \right\} \\ &= \max_{i \in \{1, \dots, n\}} \min_{j \in \{1, \dots, m\}} \left( (e^j)^T \cdot A \right) \cdot e^i \\ &= \max_{e_i} \min_{e_j} \left( (e^j)^T \cdot A \right) \cdot e^i \\ &\leq \max_x \min_{\pi} (\pi^T \cdot A) \cdot x = M_0 \end{aligned}$$

$$\begin{aligned} M^0 &= \min_{\pi} \max_x \pi^T \cdot (A \cdot x) \leq \min_{e_i} \max_{e_j} \left( (e^j)^T \cdot A \right) \cdot e^i \\ &= \min_{j \in \{1, \dots, m\}} \max_{i \in \{1, \dots, n\}} \left( (e^j)^T \cdot A \right) \cdot e^i \\ &= \min \left\{ \max \{ a_{i,j} \mid j = 1, \dots, n \} \mid i = 1, \dots, m \right\} = a^0 \end{aligned}$$

### Proof of Lemma 10.2.8

Let  $x_0$  and  $\pi_0$  be optimal strategies. Then, we can conclude:

$$M_0 = \max_x \left( \min_{\pi} (\pi^T \cdot A) \cdot x \right) = \min_{\pi} (\pi^T \cdot A) \cdot x_0$$

$$\leq \pi_0^T \cdot A \cdot x_0 \leq \max_x \pi_0^T \cdot A \cdot x = M^0$$

$$\Rightarrow M_0 \leq M^0$$

Altogether, we obtain:  $a_0 \leq M_0 \leq M^0 \leq a^0$

### 10.3 Games and Linear Programming

- In what follows, we provide methods that generate optimal strategies for two-person games
- These methods are based on the principles of Linear Programming
- Consequently, at first, we provide an LP-problem definition

### Preliminary work I

#### 10.3.1 Lemma

Assume  $x \in S^{(n)}$  and  $M \in \mathbb{R}$ :

Then, it holds:  $A \cdot x \geq M \cdot \underbrace{(1, \dots, 1)}_{m \text{ times}} \Leftrightarrow \min_{\pi} \pi^T \cdot A \cdot x \geq M$

### Proof of Lemma 10.3.1

$$A \cdot x \geq M \cdot \underbrace{(1, \dots, 1)}_{m \text{ times}}$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^T \cdot A \cdot x \geq \pi^T \cdot M \cdot (1, \dots, 1), \text{ with } \sum_{j=1}^m \pi_j = 1$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^T \cdot A \cdot x \geq M \cdot \sum_{j=1}^m \pi_j$$

$$\Leftrightarrow \forall \pi \in S^{(m)}: \pi^T \cdot A \cdot x \geq M \Leftrightarrow \min_{\pi} \pi^T \cdot A \cdot x \geq M$$

## Preliminary work II

### 10.3.2 Lemma:

Assume  $x \in S^{(n)}$  and  $M \in \mathbb{R}$ :

Then, it holds:  $\pi^T \cdot A \leq M \cdot \underbrace{(1, \dots, 1)}_{n \text{ times}} \Leftrightarrow \max_x \pi^T \cdot A \cdot x \leq M$

## Proof of Lemma 10.3.2

$$\pi^T \cdot A \leq M \cdot \underbrace{(1, \dots, 1)}_{n \text{ times}}$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \leq M \cdot (1, \dots, 1) \cdot x, \text{ with } \sum_{i=1}^n x_i = 1$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \leq M \cdot \sum_{i=1}^n x_i$$

$$\Leftrightarrow \forall x \in S^{(n)}: \pi^T \cdot A \cdot x \leq M \Leftrightarrow \max_x \pi^T \cdot A \cdot x \leq M$$

## Preliminary work III

Assume that  $x_0$  is an optimal strategy for player 1. Then, we know that it holds:  $\min_{\pi} \pi^T \cdot A \cdot x_0 = M_0$

By making use of Lemma 10.3.1, we conclude that it holds:

$$A \cdot x_0 \geq M_0 \cdot 1_{(m)}$$

Additionally, if it holds  $M_0 > 0$ , we define  $x_1 = \frac{x_0}{M_0}$  and obtain

$$A \cdot x_1 = A \cdot \frac{x_0}{M_0} \geq 1_{(m)}. \text{ Since } x_0 \geq 0 \Rightarrow x_1 \geq 0.$$

## Preliminary work IV

Assume that  $\pi_0$  is an optimal strategy for player 2. Then, we know  $\max_x (\pi_0)^T \cdot A \cdot x = M^0$

By making use of Lemma 10.3.2, we conclude that it holds:

$$(\pi_0)^T \cdot A \leq M^0 \cdot 1_{(n)}$$

Additionally, if it holds  $M^0 > 0$ , we define  $\pi_1 = \frac{\pi_0}{M^0}$  and obtain

$$(\pi_1)^T \cdot A = \left( \frac{\pi_0}{M^0} \right)^T \cdot A \leq 1_{(n)}. \text{ Since } \pi_0 \geq 0 \Rightarrow \pi_1 \geq 0.$$

## The Linear Program

We introduce the following Linear Program (P)

Minimize  $1^T \cdot x$ , s.t.  $A \cdot x \geq 1 \wedge x \geq 0$

as the LP that corresponds to the game matrix  $A$

## Observations

### 10.3.3 Lemma:

1. Let  $x$  be feasible for  $(P)$  with  $M = \frac{1}{1^T \cdot x}$ , then  $M > 0$  and for  $\tilde{x} = M \cdot x$  it holds:  $\tilde{x} \in S^{(n)} \wedge \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M$ .
2. Other way round, if  $\tilde{x} \in S^{(n)} \wedge M = \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} > 0$ , then it holds that  $x$  is feasible for  $(P)$ , with  $x = \frac{\tilde{x}}{M}$ , and  $1^T \cdot x = \frac{1}{M}$ .

## Proof of Lemma 10.3.3

1. Let  $x$  be feasible for  $(P)$  with  $M = \frac{1}{1^T \cdot x} \Rightarrow x \geq 0 \wedge A \cdot x \geq 1 \Rightarrow 1^T \cdot x > 0 \Rightarrow M = \frac{1}{1^T \cdot x} > 0$ . Let  $\tilde{x} = M \cdot x \Rightarrow A \cdot \tilde{x} = A \cdot M \cdot x = M \cdot A \cdot x \geq M \cdot 1$ .

Thus, we can apply Lemma 10.3.1  $\Rightarrow \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M$ .

Additionally, we calculate  $1^T \cdot \tilde{x} = 1^T \cdot M \cdot x = 1^T \cdot \frac{1}{1^T \cdot x} \cdot x = 1$

and obviously  $\tilde{x} \geq 0$ . Thus,  $\tilde{x} \in S^{(n)}$ .

## Proof of Lemma 10.3.3

2. Let  $\tilde{x} \in S^{(n)} \wedge \min_{\pi \in S^{(m)}} \pi^T \cdot A \cdot \tilde{x} \geq M > 0 \Rightarrow \tilde{x} \geq 0 \wedge 1^T \cdot \tilde{x} = 1$   
Since Lemma 10.3.1, it additionally holds  $A \cdot \tilde{x} \geq M \cdot (1, \dots, 1)$

$\Rightarrow$  We define  $x = \frac{\tilde{x}}{M} \Rightarrow A \cdot x = \frac{A \cdot \tilde{x}}{M} \geq \frac{M}{M} = 1 \Rightarrow A \cdot x \geq 1$

Since  $M > 0$ , it holds that  $x = \frac{\tilde{x}}{M} \geq 0$ .

Consequently,  $x$  is feasible for  $(P)$ , with  $1^T \cdot x = 1^T \cdot \frac{\tilde{x}}{M} = \frac{1}{M}$ .

## and for optimal solutions of the LP...

### 10.3.4 Lemma

Let  $x_0$  be an optimal solution of LP. Then it holds:

$$1^T \cdot x_0 = \frac{1}{M_0}$$

## Proof of Lemma 10.3.4

At first, we show that  $\frac{1}{M_0}$  is a lower bound for the objective function  $1^T \cdot x$  if the problem is solvable. Let  $x$  be some feasible

solution for the LP. Furthermore, let  $\tilde{x} = \frac{x}{M(x)}$ , with

$$M(x) = 1^T \cdot x. \text{ Thus, we conclude } A \cdot \tilde{x} = \frac{A \cdot x}{M(x)} \geq \frac{1}{M(x)}$$

$$\Rightarrow \min_{\pi} \pi^T \cdot A \cdot \tilde{x} \geq \frac{1}{M(x)}. \text{ It holds: } M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x$$

$$\Rightarrow \frac{1}{M(x)} = \frac{1}{1^T \cdot x} \leq M_0 \Leftrightarrow 1^T \cdot x \geq \frac{1}{M_0}.$$

## Proof of Lemma 10.3.4

Now, other way round, let  $x_0$  be an optimal solution to LP (P) as defined above.

Consider now  $M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x = \min_{\pi} \pi^T \cdot A \cdot \tilde{x}$ , with  $\tilde{x} \in S^{(n)}$ . Thus, by making use of Lemma 10.3.3(2), we know that

$$M_0 = \frac{1}{1^T \cdot \tilde{x}} \text{ and } x_1 = \tilde{x} \cdot \frac{1}{M_0} \text{ is a feasible solution to LP. Since } x_0$$

is an optimal solution to LP, we obtain  $1^T \cdot x_0 \leq 1^T \cdot x_1$

$$= \underbrace{1^T \cdot \tilde{x}}_{=1, \text{ since } \tilde{x} \in S^{(n)}} \cdot \frac{1}{M_0} = \frac{1}{M_0}.$$

$$\text{Consequently, we get: } 1^T \cdot x_0 \leq \frac{1}{M_0} \wedge 1^T \cdot x_0 \geq \frac{1}{M_0} \Rightarrow 1^T \cdot x_0 = \frac{1}{M_0}.$$

## Consequence

Obviously, if  $x_0$  is an optimal solution to the LP, we know that

$$\tilde{x} = \frac{x_0}{1^T \cdot x_0} = x_0 \cdot M \text{ is a feasible strategy and we consider}$$

$$A \cdot \tilde{x} = A \cdot \frac{x_0}{1^T \cdot x_0} = \frac{A \cdot x_0}{1^T \cdot x_0} \geq \frac{(1, \dots, 1)}{1^T \cdot x_0} = (M, \dots, M).$$

Since  $x$  is minimally chosen, we get a maximal  $M$  and therefore  $(M, \dots, M)$  is maximized. Consequently, we just maximize

$\min_{\pi} \pi^T \cdot A \cdot x$ . Unfortunately, if  $A$  is defined that way that  $M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x \leq 0$ , the problem is not solvable.

## Main Cognition

### 10.3.5 Theorem:

Assuming  $M_0 > 0$ . Then,  $x_0$  is an optimal solution to the LP if and only if  $\tilde{x}_0 = x_0 \cdot M_0 = \frac{x_0}{1^T \cdot x_0}$  is an optimal strategy for player 1.

## Proof of Theorem 10.3.5

$\Rightarrow$

Let  $x_0$  be an optimal solution to LP (P). Then,  $\tilde{x}_0 = x_0 \cdot M_0 \in S^{(n)}$ , with  $M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x \geq \min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0$ . Other way round, we get:

$$\begin{aligned} \min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 &= \min_{\pi} \pi^T \cdot A \cdot x_0 \cdot M_0 \\ &= \min_{\pi} \pi^T \cdot \underbrace{A \cdot x_0}_{\geq 1, \text{ since } x_0 \text{ is feasible for LP}} \cdot \frac{1}{1^T \cdot x_0} \\ &\geq \min_{\pi} \pi^T \cdot \frac{(1, \dots, 1)^T}{1^T \cdot x_0} = \underbrace{\pi^T \cdot (1, \dots, 1)^T}_{=1, \text{ since } \pi \in S^{(m)}} \cdot \frac{1}{1^T \cdot x_0} = \frac{1}{1^T \cdot x_0} = M_0 \end{aligned}$$

## Proof of Theorem 10.3.5

$\Rightarrow$

Consequently, we obtain

$$\min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 \leq M_0 \wedge \min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 \geq M_0$$

$$\Rightarrow \min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 = M_0$$

$$\Rightarrow \tilde{x}_0 \in S^{(n)} \text{ is optimal!}$$

## Proof of Theorem 10.3.5

$\Leftarrow$

Let  $\tilde{x}_0 \in S^{(n)}$  be an optimal strategy. Then  $\min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 = M_0$ .

Thus, we make use of Lemma 10.3.3, and obtain

$$A \cdot \tilde{x}_0 \geq M_0. \text{ Consider } x_0 = \frac{\tilde{x}_0}{M_0}.$$

$$\text{It holds: } A \cdot x_0 = A \cdot \frac{\tilde{x}_0}{M_0} \geq \frac{M_0}{M_0} = 1.$$

Thus,  $x_0$  is feasible for LP. We calculate

$$1^T \cdot x_0 = 1^T \cdot \frac{\tilde{x}_0}{M_0} = \frac{1^T \cdot \tilde{x}_0}{M_0} = \frac{1}{M_0}.$$

### Proof of Theorem 10.3.5

We know that  $\frac{1}{M_0}$  is a lower bound for  $1^T \cdot x$   
if  $x$  is a feasible solution for LP (P).

Consequently, we have shown that  $x_0$  is an optimal solution.

This completes the proof.

### The dual program

➤ The LP introduced above has the following dual

Maximize  $1^T \cdot \pi$ ,

s.t.

$$\pi^T \cdot A \leq 1^T \wedge \pi \geq 0$$

⇒ Obviously,  $\pi = 0$  is a feasible solution to (D)

### Further consequences

#### 10.3.6 Corollary

*The following propositions are equivalent*

1. (P) has a feasible solution
2. (P) has an optimal solution
3. (D) has an optimal solution
4.  $M_0 > 0$

### Proof of Corollary 10.3.6

$1 \Leftrightarrow 4$ :

Let  $x$  be a feasible solution to (LP). Then, we have

$$A \cdot x \geq 1 \text{ and thus } M = \min_{\pi} \pi^T \cdot A \cdot x \geq 1 \Rightarrow$$

$$M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x \geq M > 0.$$

Other way round, if  $M_0 = \max_x \min_{\pi} \pi^T \cdot A \cdot x > 0$ .

Thus, it exists  $\tilde{x}_0 : M_0 = \min_{\pi} \pi^T \cdot A \cdot \tilde{x}_0 \Rightarrow A \cdot \tilde{x}_0 \geq M_0$ .

We introduce:  $x_0 = \tilde{x}_0 \cdot \frac{1}{M_0} \geq 0$

$$\Rightarrow A \cdot x_0 = A \cdot \tilde{x}_0 \cdot \frac{1}{M_0} \geq M_0 \cdot \frac{1}{M_0} = 1 \Rightarrow x_0 \text{ is feasible for (LP).}$$

### Proof of Corollary 10.3.6

$1 \Leftrightarrow 2 \Leftrightarrow 3$ :

Obviously,  $(D)$  is always solvable, e.g., by making use of  $\pi = 0$ , we have at least one feasible solution.

Thus, through Section 2.2, there remain two cases.

Either  $(D)$  is unrestricted and, therefore,  $(LP)$  not solvable or  $(D)$  and  $(LP)$  have optimal solutions.

This completes the proof.

### What to do if it holds that $M_0 \leq 0$ ?

- Obviously, if  $M_0 > 0$ , we have provided an instrument that generates optimal strategies
- But if  $M_0 \leq 0$ , nothing is won since LP (P) is obviously not solvable
- However, what can we do in such kind of situation?
- Obviously, it is matrix  $A$  that incorporates this problem. Thus, the question to be posed is how we can modify this matrix in order to ensure that  $M_0 > 0$

### Adding a constant to $A$

Let us add a constant  $C$  to all matrix entries. We consider the result of a game, i.e.,

$$\begin{aligned} \pi^T \cdot A \cdot x &= \sum_{j=1}^n \left( \sum_{i=1}^m \pi_i \cdot (a_{i,j} + C) \right) \cdot x_j = \sum_{j=1}^n \left( \sum_{i=1}^m \pi_i \cdot a_{i,j} + (\pi_i \cdot C) \right) \cdot x_j \\ &= \sum_{j=1}^n \sum_{i=1}^m x_j \cdot \pi_i \cdot a_{i,j} + C \cdot \underbrace{\sum_{j=1}^n \sum_{i=1}^m x_j \cdot \pi_i}_{=1} = \sum_{j=1}^n \sum_{i=1}^m x_j \cdot \pi_i \cdot a_{i,j} + C \end{aligned}$$

$\Rightarrow$

We obtain modified proceeds, but optimality is kept unchanged.

### Consequences

- By adding a constant, we may be able to obtain a matrix  $A_{mod}$  that fulfills  $M_0 > 0$
- The game that corresponds to the modified matrix  $A_{mod}$  has identical optimal strategies
- Consequently, we only have to retransform the resulting profits at the end of the calculation process

## What to add?

- Fortunately, we know that  $M_0$  is just raised by the value/constant  $C$  that is added to  $A$
- The problem is that  $M_0$  is unknown beforehand
- Otherwise, we just would take  $-M_0 + \varepsilon$ , with  $\varepsilon > 0$
- Consequently, we may take  $-a_0 + \varepsilon$  ( $\varepsilon > 0$ ), which is a lower bound of  $M_0$

## Further bounds to add

2. Possibility :

Sufficient conditions are :  $A \geq 0 \wedge a^0 = \min_i \max_j a_{i,j} > 0$

Consider the vector  $1^T \cdot \frac{1}{a^0} \Rightarrow A \cdot \left( 1^T \cdot \frac{1}{a^0} \right) = \frac{1}{a^0} \cdot (A \cdot 1^T) \geq \frac{1}{a^0} \cdot a^0 \cdot (1, \dots, 1)$   
 $= (1, \dots, 1) \Rightarrow 1^T \cdot \frac{1}{a^0}$  is feasible for (LP) !

$\Rightarrow$  We have to add  $\max \{ -\min a_{i,j}, -a^0 + \varepsilon \}$ , with  $\varepsilon > 0$

## Further bounds to add

3. Possibility :

Sufficient condition is :  $a_1 = \min_i \sum_{j=1}^n a_{i,j} > 0$

Consider the vector  $1^T \cdot \frac{1}{a_1} \Rightarrow A \cdot \left( 1^T \cdot \frac{1}{a_1} \right) = \frac{1}{a_1} \cdot (A \cdot 1^T) \geq \frac{1}{a_1} \cdot a_1 \cdot (1, \dots, 1)$   
 $= (1, \dots, 1) \Rightarrow 1^T \cdot \frac{1}{a_1}$  is feasible for (LP)!

$\Rightarrow$  We have to add  $-\min_i \left( \frac{1}{n} \cdot \left( \sum_{j=1}^n a_{i,j} \right) \right) + \varepsilon$ , with  $\varepsilon > 0$

## Summary

We always add:

$$\min \left\{ -a_0 + \varepsilon, -\min a_{i,j}, -a^0 + \varepsilon, -\min_i \left( \frac{1}{n} \cdot \left( \sum_{j=1}^n a_{i,j} \right) \right) + \varepsilon \right\}$$

Note that if it holds  $M_0 \gg 0$ , the value above becomes negative and matrix  $A$  is reduced

### 10.3.7 Example

- We now come back to our two introducing examples 10.1.1 and 10.1.2
- We start with example 10.1.1
- This was the simple game “rock, scissors, and paper”

### Example

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow$$

$$1. a_0 = \max_i \min_j a_{i,j} = \max\{-1, -1, -1\} = -1 \Rightarrow C = 1 + \varepsilon$$

$$2. \min_{i,j} a_{i,j} = -1 \text{ and } a^0 = \min_i \max_j a_{i,j} = \min\{1, 1, 1\} = 1 \\ \Rightarrow C = \max\{-\min_{i,j} a_{i,j}, -a^0 + \varepsilon\} = \max\{1, -1 + \varepsilon\} = 1$$

$$3. \min\left\{\frac{1}{3} \cdot 0, \frac{1}{3} \cdot 0, \frac{1}{3} \cdot 0\right\} = 0 \Rightarrow C = \varepsilon$$

### We take $C=1$

Thus, we obtain the following matrix

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \Rightarrow \tilde{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \Rightarrow \tilde{A}^T = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

Maximize  $1^T \cdot \pi \Leftrightarrow$  Minimize  $-1^T \cdot \pi$ , w.r.t.,  $\tilde{A}^T \cdot \pi \leq 1 \wedge \pi \geq 0$

We solve the dual problem by introducing slackness variables

### Calculation...

0	-1	-1	-1	0	0	0
1	1	2	0	1	0	0
1	0	1	2	0	1	0
1	2	0	1	0	0	1

### Calculation...

1	0	1	-1	1	0	0
1	1	2	0	1	0	0
1	0	1	2	0	1	0
-1	0	-4	1	-2	0	1

### Calculation...

0	0	0	-3	1	-1	0
-1	1	0	-4	1	-2	0
1	0	1	2	0	1	0
3	0	0	9	-2	4	1

### Calculation...

0	0	0	-3	1	-1	0
-1	1	0	-4	1	-2	0
1	0	1	2	0	1	0
$\frac{1}{3}$	0	0	1	$-\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

### Calculation...

0	0	0	-3	1	-1	0
$\frac{1}{3}$	1	0	0	$\frac{1}{9}$	$-\frac{2}{9}$	$\frac{4}{9}$
$\frac{1}{3}$	0	1	0	$\frac{4}{9}$	$\frac{1}{9}$	$-\frac{2}{9}$
$\frac{1}{3}$	0	0	1	$-\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

## Calculation...

1	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\frac{1}{3}$	1	0	0	$\frac{1}{9}$	$-\frac{2}{9}$	$\frac{4}{9}$
$\frac{1}{3}$	0	1	0	$\frac{4}{9}$	$\frac{1}{9}$	$-\frac{2}{9}$
$\frac{1}{3}$	0	0	1	$-\frac{2}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

## Result

First row :  $c^T - c_B^T \cdot A_B^{-1} \cdot E \Rightarrow x^T = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \pi^T$

$$M_0 = 1 - C = 1 - 1 = 0.$$

## One can analogously show...

### 10.3.8 Observation:

Assuming  $M^0 > 0$ . Then,  $\pi^0$  is an optimal solution to the dual of (LP) if and only if  $\tilde{\pi}^0 = \pi^0 \cdot M^0 = \frac{\pi^0}{1^T \cdot \pi^0}$  is an optimal strategy for player 2.

## Fundamental Theorem of matrix games

### 10.3.9 Theorem:

1. There are always optimal strategies and for each pair of optimal strategies  $x_0$  and  $\pi^0$  it holds:  $M_0 = \pi^{0T} \cdot A \cdot x_0 = M^0$ .
2. Optimal strategies  $x_0$  and  $\pi^0$  always fulfill  $\min_i e^{iT} \cdot A \cdot x_0 = M_0 = M^0 = \max_j \pi^{0T} \cdot A \cdot e^j$ .

### Proof of Theorem 10.3.9

ad 1.

By adding the value  $C$ , we obtain an optimally solvable LP, whose optimal solution always corresponds to an optimal strategy. Thus, we conclude the general solvability of matrix games.

We assume we have a pair of optimal strategies  $x_0$  and  $\pi^0$ . Then, we know that  $\frac{1}{M_0}$  and  $\frac{1}{M^0}$  are objective function values of the primal and dual

program, respectively. Thus, we conclude  $M_0 = M^0$ .

Additionally, we know:

$$\begin{aligned} M_0 &= \min_{\pi} \pi^T \cdot A \cdot x_0 \leq \pi^{0T} \cdot A \cdot x_0 \leq \max_x \pi^{0T} \cdot A \cdot x = M^0 \\ \Rightarrow M_0 &= \min_{\pi} \pi^T \cdot A \cdot x_0 = \pi^{0T} \cdot A \cdot x_0 = \max_x \pi^{0T} \cdot A \cdot x = M^0 \end{aligned}$$

### Proof of Theorem 10.3.9

ad 2.

Let  $x_0$  and  $\pi^0$  be optimal strategies. Then, we know

$$M_0 = \min_{\pi} \pi^T \cdot A \cdot x_0 = \max_x \pi_0^T \cdot A \cdot x = M^0. \text{ In addition,}$$

$$\text{we have } \min_{\pi} \pi^T \cdot A \cdot x_0 \leq \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_0.$$

Obviously, we have  $\forall \pi \in S^{(m)} : \pi = \sum_{i=1}^m \pi_i \cdot e^i$ . Thus, we obtain:

$$\min_{\pi} \pi^T \cdot A \cdot x_0 = \pi^{0T} \cdot A \cdot x_0 = \left( \sum_{i=1}^m \pi_i^0 \cdot e^i \right)^T \cdot A \cdot x_0 = \sum_{i=1}^m \pi_i^0 \cdot (e^{iT} \cdot A) \cdot x_0$$

$$\geq \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_0 \Rightarrow \min_{\pi} \pi^T \cdot A \cdot x_0 = \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_0$$

$$\text{Analogously, one can show } \max_x \pi_0^T \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi_0^T \cdot A \cdot e^j.$$

### Proof of Theorem 10.3.9

Hence, we can conclude:

$$\begin{aligned} \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_0 &= \min_{\pi} \pi^T \cdot A \cdot x_0 = M_0 = M^0 = \max_x \pi^{0T} \cdot A \cdot x \\ &= \max_{e^j \in S^{(n)}} \pi^{0T} \cdot A \cdot e^j \end{aligned}$$

Other way round, if  $\min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_1 = \max_{e^j \in S^{(n)}} \pi^{1T} \cdot A \cdot e^j$

for a pair of strategies  $x_1$  and  $\pi^1$ , we obtain

$$\begin{aligned} M_0 &= \min_{\pi} \max_x \pi^T \cdot A \cdot x \leq \max_x \pi^{1T} \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi^{1T} \cdot A \cdot e^j \\ &= \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_1 \leq \min_{\pi} \pi^T \cdot A \cdot x_1 \leq \max_x \min_{\pi} \pi^T \cdot A \cdot x = M^0 \end{aligned}$$

### Proof of Theorem 10.3.9

$$\Rightarrow \text{Since } M_0 = M^0 \Rightarrow$$

$$\begin{aligned} M_0 &= \min_{\pi} \max_x \pi^T \cdot A \cdot x = \max_x \pi^{1T} \cdot A \cdot x = \max_{e^j \in S^{(n)}} \pi^{1T} \cdot A \cdot e^j \\ &= \min_{e^j \in S^{(m)}} e^{iT} \cdot A \cdot x_1 = \min_{\pi} \pi^T \cdot A \cdot x_1 = \max_x \min_{\pi} \pi^T \cdot A \cdot x = M^0 \end{aligned}$$

### 10.3.10 Example

Example 9.1.2:

$$A = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix} \Rightarrow$$

1.  $a_0 = \max_i \min_j a_{i,j} = \max\{44, 58, 60, 59\} = 60 \Rightarrow C = -60 + \varepsilon$
2.  $\min_{i,j} a_{i,j} = 44$  and  $a^0 = \min_i \max_j a_{i,j} = \min\{72, 68, 75, 64\} = 64$   
 $\Rightarrow C = \max\{-\min_{i,j} a_{i,j}, -a^0 + \varepsilon\} = \max\{-44, -64 + \varepsilon\} = -44$
3.  $\min\left\{-\frac{1}{4} \cdot 244, -\frac{1}{4} \cdot 251, -\frac{1}{4} \cdot 279, -\frac{1}{4} \cdot 240\right\} = -60 \Rightarrow C = -60 + \varepsilon$

We take  $C = -59$

### The resulting program

$$A - 59 = \begin{pmatrix} 44 & 72 & 64 & 64 \\ 68 & 58 & 60 & 65 \\ 64 & 68 & 72 & 75 \\ 56 & 64 & 61 & 59 \end{pmatrix} - 59 = \begin{pmatrix} -15 & 13 & 5 & 5 \\ 9 & -1 & 1 & 6 \\ 5 & 9 & 13 & 16 \\ -3 & 5 & 2 & 0 \end{pmatrix} = \tilde{A}$$

$$\tilde{A}^T = \begin{pmatrix} -15 & 9 & 5 & -3 \\ 13 & -1 & 9 & 5 \\ 5 & 1 & 13 & 2 \\ 5 & 6 & 16 & 0 \end{pmatrix}$$

### Optimal solution

By using  $\tilde{A}^T = \begin{pmatrix} -15 & 9 & 5 & -3 \\ 13 & -1 & 9 & 5 \\ 5 & 1 & 13 & 2 \\ 5 & 6 & 16 & 0 \end{pmatrix}$ ,

we obtain the optimal solution

$$\tilde{x} = \left(0, \frac{1}{5}, 0, \frac{1}{5}\right)^T \wedge \tilde{\pi} = \left(0, \frac{1}{6}, 0, \frac{7}{30}\right)^T$$

### Optimal solution – objective function value

$$\begin{aligned} \Rightarrow \tilde{\pi}^T \cdot \tilde{A} \cdot x &= \left(0, \frac{1}{6}, 0, \frac{7}{30}\right) \cdot \begin{pmatrix} -15 & 13 & 5 & 5 \\ 9 & -1 & 1 & 6 \\ 5 & 9 & 13 & 16 \\ -3 & 5 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} \\ &= \left(\frac{9}{6} - \frac{21}{30} \quad -\frac{1}{6} + \frac{35}{30} \quad \frac{1}{6} + \frac{14}{30} \quad 1\right) \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} \\ &= \left(\frac{4}{5} \quad 1 \quad \frac{19}{30} \quad 1\right) \cdot \begin{pmatrix} 0 \\ \frac{1}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix} = \frac{2}{5} \end{aligned}$$

## Results

$$\tilde{M}_0 = \frac{1}{2/5} = \frac{5}{2} = \frac{1}{(1 \ 1 \ 1 \ 1)^T \cdot \tilde{x}} = \frac{1}{\frac{1}{5} + \frac{1}{5}} \wedge$$

$$\tilde{M}^0 = \frac{1}{2/5} = \frac{5}{2} = \frac{1}{(1 \ 1 \ 1 \ 1)^T \cdot \tilde{\pi}} = \frac{1}{\frac{1}{6} + \frac{7}{30}} = \frac{1}{\frac{12}{30}} = \frac{1}{\frac{2}{5}}$$

## Transformation

With  $\tilde{M}_0 = \frac{5}{2}$  we get

$$\tilde{x}_0 = \tilde{x} \cdot \frac{5}{2} = \left(0 \ \frac{1}{5} \ 0 \ \frac{1}{5}\right)^T \cdot \frac{5}{2} = \left(0 \ \frac{1}{2} \ 0 \ \frac{1}{2}\right)^T \wedge$$

$$\tilde{\pi}^0 = \tilde{\pi} \cdot \frac{5}{2} = \left(0 \ \frac{1}{6} \ 0 \ \frac{7}{30}\right)^T \cdot \frac{5}{2} = \left(0 \ \frac{5}{12} \ 0 \ \frac{7}{12}\right)^T$$

$$\Rightarrow M_0 = M^0 = \frac{5}{2} + 59 = 61.5$$

$$\Rightarrow a_0 = 60 < M_0 = 61.5 = M^0 < a^0 = 64$$