4 Hitchcock Transportation Problem

The balanced transportation problem is defined as follows:

$c_{i,j}$: Delivery costs for each product unit that is transported from supplier i to customer j

a_i : Total supply of $i = 1, \dots, m$

b_j : Total demand of $j = 1, \dots, n$

$x_{i,j}$: Quantity that supplier $i = 1, \dots, m$ delivers to the customer $j = 1, \dots, n$

(P) Minimize $c^T \cdot x$

$$\text{s.t.} \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \dots & \dots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} \cdot x = \begin{pmatrix} a_1 \\ \dots \\ \dots \\ a_m \\ b \end{pmatrix}$$

$$x = \left(x_{1,1}, \dots, x_{1,j}, \dots, x_{1,n}, \dots, x_{i,1}, \dots, x_{i,n} \dots, x_{m,1}, \dots, x_{m,n}\right)^T \geq 0$$

The dual problem

- Thus, we obtain as the dual problem

$$(D)\ \text{Maximize} \sum_{i=1}^{m} a_i \cdot \pi_i + \sum_{j=1}^{n} b_j \cdot \pi_{m+j} = \sum_{i=1}^{m} a_i \cdot \alpha_i + \sum_{j=1}^{n} b_j \cdot \beta_j \ \text{ s.t.}$$

$$\begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \ldots & & E_n \\ & & \ldots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \pi \le \begin{pmatrix} c_{1,1} \\ \ldots \\ c_{i,1} \\ \ldots \\ c_{m,n} \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1_n & & & & E_n \\ & 1_n & & & E_n \\ & & \ldots & & E_n \\ & & \ldots & & E_n \\ & & & 1_n & E_n \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \le \begin{pmatrix} c_{1,1} \\ \ldots \\ c_{i,1} \\ \ldots \\ c_{m,n} \end{pmatrix}$$

i.e.

$$\forall i \in \{1,\ldots,m\} : \forall j \in \{1,\ldots,n\} : \alpha_i + \beta_j \le c_{i,j}$$

$$\forall i \in \{1,\ldots,m\} : \alpha_i \text{ free} \wedge \forall j \in \{1,\ldots,n\} : \beta_j \text{ free}$$

4.1 Using the Simplex Algorithm

- Relevant costs are calculated as follows

$$\forall i \in \{1,\ldots,m\} : \forall j \in \{1,\ldots,n\} : \overline{c}_{i,j} = c_{i,j} - \left(\pi^T \cdot A\right)_{(i-1)\cdot n+j} = c_{i,j} - \left(A^T \cdot \pi\right)_{(i-1)\cdot n+j}$$
$$= c_{i,j} - \alpha_i - \beta_j$$

- Observation: Consider the matrix *A*

$$A = \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \ldots & \ldots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix}$$

Transportation matrix

$$A = \begin{pmatrix} 1_n^T & & & & \\ & 1_n^T & & & \\ & & \cdots & \cdots & \\ & & & & 1_n^T \\ E_n & E_n & E_n & E_n & E_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \\ \\ a_m \\ \hat{A} \end{pmatrix} = \begin{pmatrix} a_1 \\ \\ \\ a_m \\ \hat{a}_1 \\ \cdots \\ \hat{a}_n \end{pmatrix} \in IR^{(m+n)\times(m\cdot n)}$$

$$\text{Obviously, it holds}: \sum_{i=1}^{m} a_i - \sum_{i=1}^{n} \hat{a}_i = 0 \Leftrightarrow \hat{a}_n = \sum_{i=1}^{m} a_i - \sum_{i=1}^{n-1} \hat{a}_i$$

Consequences

- Thus, we obviously can skip the last row of matrix *A*
- Note that this does not have any impact on the problem solvability since there is direct dependency between the *a*- and the *b*-vector, too
- Specifically, it holds:

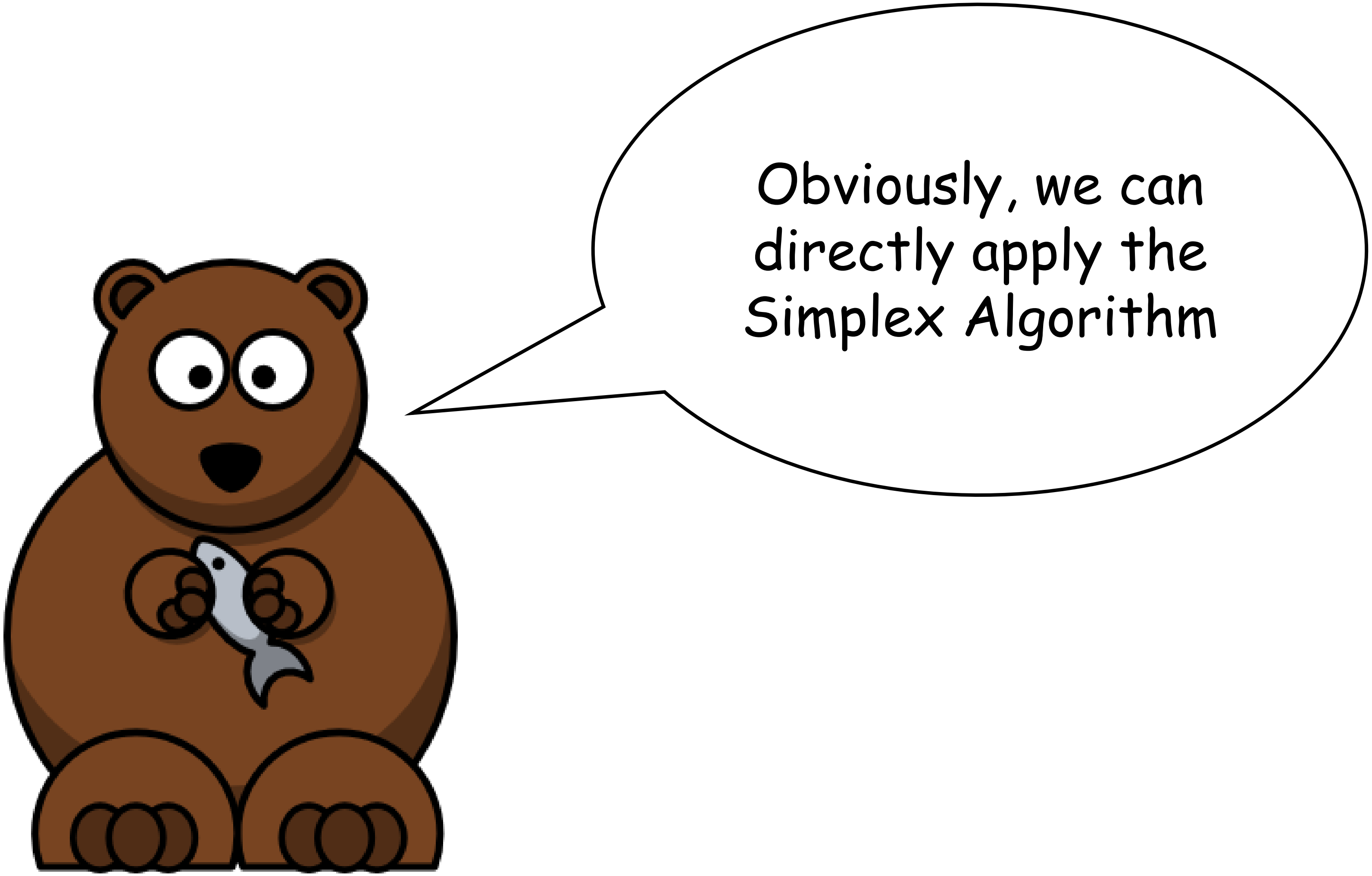
$$\boxed{\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \Leftrightarrow \sum_{i=1}^{m} a_i - \sum_{j=1}^{n-1} b_j = b_n}$$

Example

- We consider the following constellation:

$$a = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}; b = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 3 \end{pmatrix}; c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 3 \end{pmatrix}$$

What is to do?



Finding a feasible solution (1)

We add one slack variable per row that equals the right-hand side and has an objective function coefficient of one (comparable to the Two-Phase Method)

0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (2)

−25	0	0	0	0	0	0	−2	−2	−2	−1	−2	−2	−2	−1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (3a)

$$
\begin{array}{c|cccccc:cccccccccccc}
-25 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -1 & -2 & -2 & -2 & -1 \\
\hline
3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & (1) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}
$$

Finding a feasible solution (3b)

−21	0	0	0	2	0	0	0	−2	−2	−1	0	−2	−2	−1	0	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (4a)

−21	0	0	0	2	0	0	0	−2	−2	−1	0	−2	−2	−1	0	−2	−2	−1
1	1	0	0	−1	0	0	0	(1)	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (4b)

−19	2	0	0	0	0	0	0	0	0	1	−2	−2	−2	−1	−2	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	−1	0	0	1	1	0	0	0	−1	−1	1	1	0	0	1	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (5a)

−19	2	0	0	0	0	0	0	0	0	1	−2	−2	−2	−1	−2	−2	−2	−1
1	1	0	0	−1	0	0	0	1	1	1	−1	0	0	0	−1	0	0	0
5	0	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	(1)	0	0	0	1	0	0	0
2	−1	0	0	1	1	0	0	0	−1	−1	1	1	0	0	1	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (5b)

-15	2	0	0	2	0	0	2	0	0	1	0	-2	-2	-1	0	-2	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	1	0	-1	0	0	-1	0	0	0	0	1	1	1	-1	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	-1	0	0	0	1	0	-1	0	-1	-1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (6a)

−15	2	0	0	2	0	0	2	0	0	1	0	−2	−2	−1	0	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	1	0	−1	0	0	−1	0	0	0	0	1	1	1	−1	0	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	(1)	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (6b)

−15	0	0	0	2	2	0	0	0	−2	−1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	1	1	0	−1	−1	0	0	0	1	1	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (7a)

−15	0	0	0	2	2	0	0	0	−2	−1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	(1)	1	0	0	0	0	0	0	0	0
3	1	1	0	−1	−1	0	0	0	1	1	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
0	−1	0	0	0	1	0	−1	0	−1	−1	0	1	0	0	0	1	0	0
6	0	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0

Finding a feasible solution (7b)

-9	2	0	0	2	2	0	2	2	0	1	0	0	-2	-1	0	0	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	-1	-1	0	-1	-1	0	0	0	0	1	1	-1	-1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	-1	0	0	0	0	1	-1	-1	0	-1	0	0	1	0	0	0	1	0

Finding a feasible solution (8a)

−9	2	0	0	2	2	0	2	2	0	1	0	0	−2	−1	0	0	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	(1)	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0

Finding a feasible solution (8b)

−9	2	2	0	0	0	0	0	0	0	1	0	0	0	1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	−1	0	1	1	1	0	0	0	−1	0	0	0	−1	1	1	1	0

Finding a feasible solution (9a)

−9	2	2	0	0	0	0	0	0	0	1	0	0	0	1	−2	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
0	0	1	0	−1	−1	0	−1	−1	0	0	0	0	1	1	−1	−1	0	0
6	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	(1)	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
3	−1	−1	0	1	1	1	0	0	0	−1	0	0	0	−1	1	1	1	0

Finding a feasible solution (9b)

−5	2	2	0	2	0	0	2	0	0	1	2	0	0	1	0	−2	−2	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
2	0	1	0	0	−1	0	0	−1	0	0	1	0	1	1	0	−1	0	0
4	0	0	1	−1	0	0	−1	0	0	0	−1	0	0	0	0	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
1	−1	−1	0	0	1	1	−1	0	0	−1	−1	0	0	−1	0	1	1	0

Finding a feasible solution (10a)

-5	2	2	0	2	0	0	2	0	0	1	2	0	0	1	0	-2	-2	-1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
2	0	1	0	0	-1	0	0	-1	0	0	1	0	1	1	0	-1	0	0
4	0	0	1	-1	0	0	-1	0	0	0	-1	0	0	0	0	1	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0
1	-1	-1	0	0	1	1	-1	0	0	-1	-1	0	0	-1	0	(1)	1	0

Finding a feasible solution (10b)

−3	0	0	0	2	2	2	0	0	0	−1	0	0	0	−1	0	0	0	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0
3	1	1	1	−1	−1	−1	0	0	0	1	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
1	−1	−1	0	0	1	1	−1	0	0	−1	−1	0	0	−1	0	1	1	0

Finding a feasible solution (11a)

−3	0	0	0	2	2	2	0	0	0	−1	0	0	0	−1	0	0	0	−1
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	0	1	−1	−1	0	−1	0	0	1	0	0	0	1	0
3	1	1	1	−1	−1	−1	0	0	0	1	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	(1)	1	1	0	1	0	0	−1	0
1	−1	−1	0	0	1	1	−1	0	0	−1	−1	0	0	−1	0	1	1	0

Finding a feasible solution (11b)

−1	1	1	0	2	2	1	1	1	0	0	1	1	0	0	0	0	−1	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	1	−1	−1	0	−1	−1	0	0	−1	−1	0	0	0	0	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (12a)

−1	1	1	0	2	2	1	1	1	0	0	1	1	0	0	0	0	−1	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
1	0	0	1	−1	−1	0	−1	−1	0	0	−1	−1	0	0	0	0	1	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
2	1	1	0	0	0	−1	1	1	0	1	1	1	0	1	0	0	−1	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (12b)

-0	1	0	0	2	2	2	1	1	1	0	0	0	0	-1	0	0	0	-1
1	0	-1	0	0	0	1	0	0	1	0	-1	-1	0	-1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	1	1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (13a)

−0	1	0	0	2	2	2	1	1	1	0	0	0	0	−1	0	0	0	−1
1	0	−1	0	0	0	1	0	0	1	0	−1	−1	0	−1	0	0	1	0
5	1	1	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
0	0	1	1	−1	−1	−1	−1	−1	−1	0	0	0	0	(1)	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (13b)

-0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	-1	-1	0	-1	-1	0	0	-1	-1	0	0	0	0	1	1
5	1	0	-1	1	1	1	1	1	1	0	1	1	1	0	0	0	0	-1
0	0	1	1	-1	-1	-1	-1	-1	-1	0	0	0	0	1	0	0	0	1
2	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0	0
3	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
3	0	0	0	0	1	0	0	1	0	0	0	1	0	0	0	1	0	0

Finding a feasible solution (14)

$$
\begin{array}{c|cccccc:cccccccccccc}
-0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 0 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

$$\Rightarrow x_{1,1} = 0, x_{1,2} = 0, x_{1,3} = 0, x_{1,4} = 3, x_{2,1} = 0, x_{2,2} = 0, x_{2,3} = 5, x_{2,4} = 0,$$
$$x_{3,1} = 2, x_{3,2} = 3, x_{3,3} = 1, x_{3,4} = 0$$

Feasible solution

$$x_{1,1} = 0, x_{1,2} = 0, x_{1,3} = 0, x_{1,4} = 3,$$
$$x_{2,1} = 0, x_{2,2} = 0, x_{2,3} = 5, x_{2,4} = 0,$$
$$x_{3,1} = 2, x_{3,2} = 3, x_{3,3} = 1, x_{3,4} = 0,$$

$$\text{i.e., } x = \begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 5 & 0 \\ 2 & 3 & 1 & 0 \end{pmatrix}$$

What is to do?

Optimizing – Phase II

0	3	3	1	2	1	2	2	3	4	5	6	3
1	–1	–1	0	0	–1	–1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	–1
0	–1	–1	–1	0	0	0	0	[1]	0	0	0	1
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-6 & 1 & 1 & -1 & 0 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 3 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-16 & -1 & -1 & -3 & 0 & -1 & 0 & 0 & 3 & 4 & 5 & 6 & 5 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

−16	2	2	0	0	−1	0	0	0	4	5	6	2
1	−1	−1	0	0	−1	−1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	−1
0	−1	−1	−1	0	0	0	0	[1]	0	0	0	1
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-24 & -2 & 2 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 5 & 6 & 2 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & -2 & -3 & 0 & 0 & -5 & -5 & 0 & 0 & 0 & 0 & 6 & 2 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0
\end{array}
$$

Optimizing – Phase II – Preparation

$$
\begin{array}{c|cccc:cccc:cccc}
-45 & 4 & 3 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -4 \\
\hline
1 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 1 \\
5 & 1 & 1 & 1 & 0 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & -1 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 1a

−45	4	3	0	0	1	1	0	0	0	0	0	−4
1	−1	−1	0	0	−1	−1	0	0	0	0	[1]	1
5	1	1	1	0	1	1	[1]	0	0	0	0	−1
0	−1	−1	−1	0	0	0	0	[1]	0	0	0	(1)
2	1	0	0	0	1	0	0	0	[1]	0	0	0
3	1	1	1	[1]	0	0	0	0	0	0	0	0
3	0	1	0	0	0	1	0	0	0	[1]	0	0

Optimizing – Phase II – Step 1b

$$
\begin{array}{r|cccc:cccc:cccc}
-45 & 0 & -1 & -4 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & (1) \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 1c

$$
\begin{array}{c|cccc:cccc:cccc}
-45 & 0 & -1 & -4 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & 1 & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 2a

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-45 & 0 & -1 & -4 & 0 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & (1) & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 2b

$$
\begin{array}{r|cccc:cccc:cccc}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & (1) & 1 & [1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 2c

$$
\begin{array}{c|cccc:cccc:cccc}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & \boxed{1} & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & \boxed{1} & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\
3 & 1 & \boxed{1} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 3a

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-42 & 1 & 0 & -3 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & (1) & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
3 & 1 & [1] & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0
\end{array}
$$

Optimizing – Phase II – Step 3b

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & (1) & 0 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 3c

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & [1] & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4a

$$
\begin{array}{c|cccc|cccc|cccc}
-39 & 1 & 0 & 0 & 1 & -2 & -2 & 0 & 1 & 0 & 0 & 3 & 0 \\
\hline
1 & 0 & 0 & [1] & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 \\
5 & 0 & 0 & 0 & 0 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & (1) & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
2 & 1 & [1] & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4b

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-35 & 3 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 2 & 0 & 3 & 0 \\
\hline
3 & 1 & 0 & [1] & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 & [1] & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & (1) & 0 & 0 & 0 & [1] & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 4c

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-35 & 3 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 2 & 0 & 3 & 0 \\
\hline
3 & 1 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 & \boxed{1} & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \boxed{1} \\
2 & 1 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & \boxed{1} & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 5a

$$
\begin{array}{r|rrrr:rrrr:rrrr}
-35 & 3 & 0 & 0 & 1 & 0 & -2 & 0 & 1 & 2 & 0 & 3 & 0 \\
\hline
3 & 1 & 0 & [1] & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
3 & -1 & 0 & 0 & 0 & 0 & 1 & [1] & 1 & -1 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & (1) & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 5b

$$
\begin{array}{r|rrrr|rrrr|rrrr}
-35 & 3 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \\
\hline
3 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & [1] & 0 & 1 & 0 & (1) & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimizing – Phase II – Step 5c

$$
\begin{array}{c|cccc|cccc|cccc}
-35 & 3 & 2 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 \\
\hline
3 & 1 & 1 & [1] & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -1 & -1 & 0 & -1 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & [1] \\
2 & 1 & 0 & 0 & 0 & [1] & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & [1] & 0 & 1 & -1 & 0 & -1 & 0 \\
3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & [1] & 1 & 0
\end{array}
$$

Optimal solution

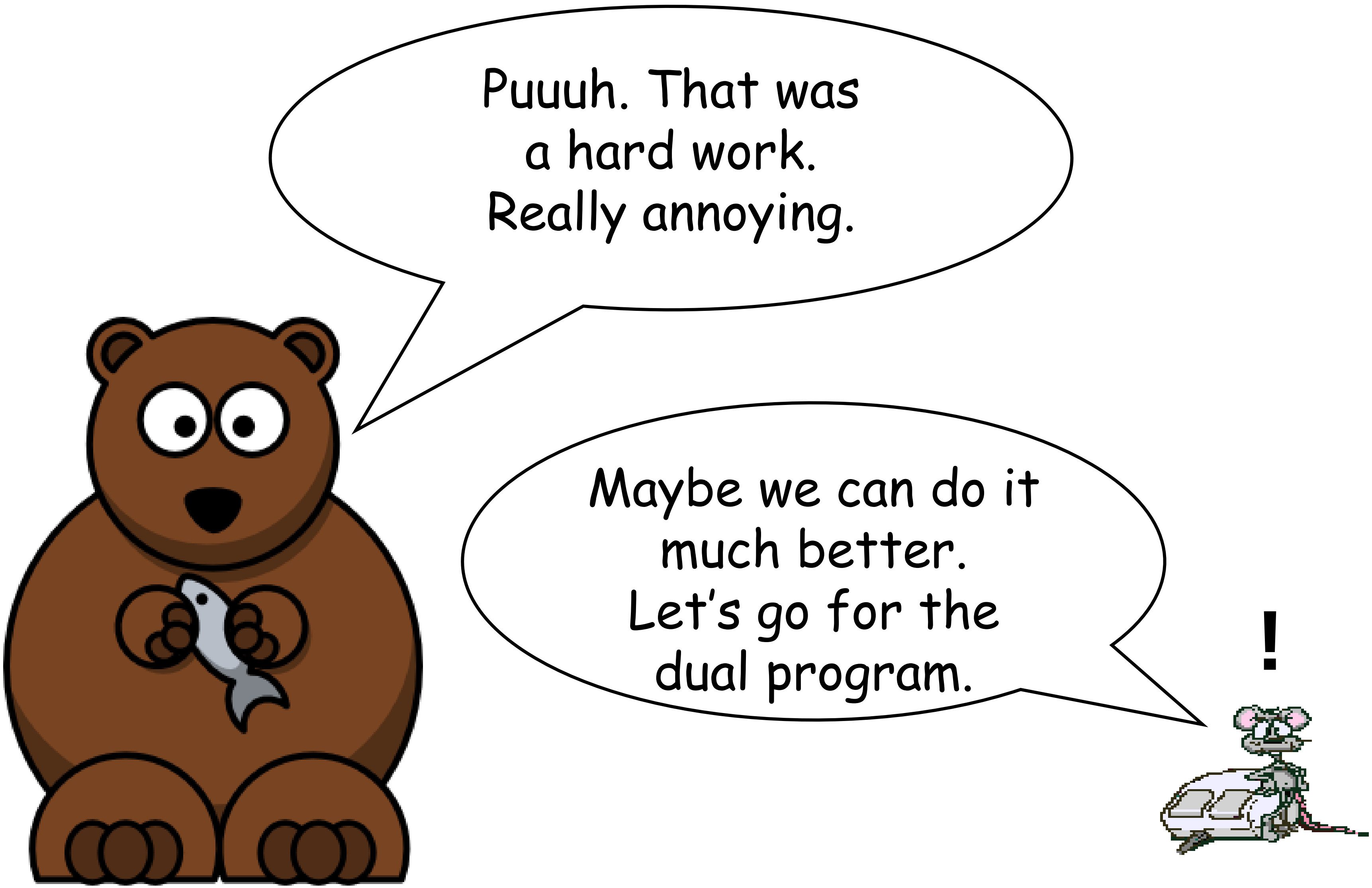
$$x_{1,1}=0, x_{1,2}=0, x_{1,3}=3, x_{1,4}=0,$$

$$x_{2,1}=2, x_{2,2}=0, x_{2,3}=3, x_{2,4}=0,$$

$$x_{3,1}=0, x_{3,2}=3, x_{3,3}=0, x_{3,4}=3,$$

$$\text{i.e., } x=\begin{pmatrix} 0 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \end{pmatrix}$$

And now?


Puuuh. That was a hard work. Really annoying.
Maybe we can do it much better. Let's go for the dual program.
!

4.2 The MODI Algorithm

- As we have seen already, the reduced costs are easy to compute for the Transportation Problem
- Thus, in what follows, we analyze them more in detail
- Here, a direct connection to the dual program is used

Calculating reduced costs

It holds:

$$\bar{c}^T = c^T - c_B^T \cdot A_B^{-1} \cdot A = c^T - \pi^T \cdot A \Rightarrow \bar{c}_{i,j} = c_{i,j} - \alpha_i - \beta_j$$

Let us assume: $\forall (i,j) \in B : \bar{c}_{i,j} = c_{i,j} - \alpha_i - \beta_j = 0 \wedge x$ bfs

$$\Rightarrow Z(x) = \sum_{(i,j)\in B} c_{i,j} \cdot x_{i,j} = \sum_{(i,j)\in B} \left(\alpha_i + \beta_j\right) \cdot x_{i,j} = \sum_{(i,j)\in B} \left(\alpha_i \cdot x_{i,j} + \beta_j \cdot x_{i,j}\right) =$$

$$\sum_i \sum_{j:(i,j)\in B} \alpha_i \cdot x_{i,j} + \sum_j \sum_{i:(i,j)\in B} \beta_j \cdot x_{i,j} = \sum_i \alpha_i \cdot \left(\sum_{j:(i,j)\in B} x_{i,j} \right) + \sum_j \beta_j \cdot \left(\sum_{i:(i,j)\in B} x_{i,j} \right) =$$

$$\sum_i \alpha_i \cdot \left(a_i\right) + \sum_j \beta_j \cdot \left(b_j\right) = b^T \cdot \pi$$

if π is feasible $\Rightarrow x, \pi$ are optimal!

(Later we will see that this procedure is a direct application of the Theorem of Complementary Slackness.)

Basic structure of the algorithm

- Start with a primal solution that is based on a basis *B*
- Generate a corresponding dual solution. This solution is characterized by the fact that whenever *(i,j)* belongs to basis *B*, the respective entries α_i and β_j are defined so that $\alpha_i+\beta_j=c_{i,j}$ holds
- As long as *(i,j)* exists with $\alpha_i+\beta_j>c_{i,j}$, find a cyclical exchange flow that reduces either α_i or β_j

Solution representation: bipartite graph

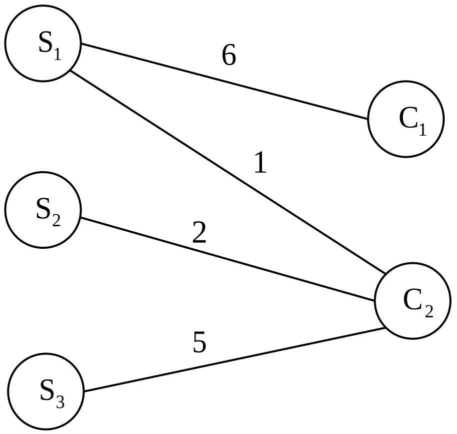
Let $x = (x_{1,1}, \dots, x_{1,n}, \dots, x_{m,1}, \dots, x_{m,n})$ be a solution for a balanced Transportation problem with m suppliers $(S_1, \dots, S_m)$ and n customers $(C_1, \dots, C_n)$. We can represent this solution as a bipartite (undirected) graph:

Example: $x_{i,j} = \begin{pmatrix} 6 & 0 & 0 \\ 1 & 2 & 5 \end{pmatrix}$

$$V = \{S_1, \dots, S_m, C_1, \dots, C_n\}$$

$$E = \{c \in \{S_1, \dots, S_m\} \times \{C_1, \dots, C_n\} : c = (S_i, C_j) \Rightarrow x_{i,j} \neq 0 \text{ for } i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$$

$$c(e) = c((S_i, C_j)) = x_{ij}$$



Cognition: structure of basic solutions

4.2.1 **Lemma**

A feasible solution for the Transportation problem is a basic feasible solution, if and only if the corresponding bipartite graph is acyclic (the undirected arcs can be used in both directions but only once).

4.2.2 Remark

If we have a basic feasible solution, we do not have more than $m + n - 1$ variables not equal to zero (as this is the number of linear independent rows in the matrix). If we have only $k < m + n - 1$ variables not equal to zero, then we insert $m + n - 1 - k$ additional edges into the bipartite graph with costs zero such that it is acyclic. This is possible since we have $m + n$ vertices and a cycle needs a node degree of 2 (but each edge connects two nodes) and therefore $2 \cdot \frac{(n+m)}{2} = n + m$ edges. Hence, with $m + n - 1$ edges and $m + n$, we can always avoid a cycle

Proof of Lemma 4.2.1

- We consider a basic feasible solution x and assume that the corresponding bipartite graph possesses a cycle
- Without losing the generality such a cycle is $S_1 \mapsto C_1 \mapsto S_2 \mapsto C_2 \mapsto \cdots \mapsto S_r \mapsto C_r \mapsto S_1$ (in order to obtain this, we can renumber customers and suppliers accordingly)
- The edge (S_i, C_i) corresponds to the following column of the LP: $\begin{pmatrix} e_S^i \\ e_C^i \end{pmatrix}$ such that $e_S^i \in \mathbb{R}^m$, $e_C^i \in \mathbb{R}^n$ are the ith unit vectors
- We consider the following linear combination of the defined column vectors with the following scalar values
- $\forall i \in \{1, \dots, r\}$: $\lambda_i = 1$ and $\forall i \in \{1, \dots, r-1\}$: $\tilde{\lambda}_i = -1$
- Moreover, we set $\lambda = -1$

Proof of Lemma 4.2.1

- We calculate

$$
\begin{aligned}
&\sum_{i=1}^{r} \lambda_i (e_S^i, e_C^i) + \sum_{i=1}^{r-1} \tilde{\lambda}_i (e_S^{i+1}, e_C^i) + \lambda (e_S^1, e_C^r) \\
&= \sum_{i=1}^{r} (e_S^i, e_C^i) - \sum_{i=1}^{r-1} (e_S^{i+1}, e_C^i) - (e_S^1, e_C^r) \\
&= \sum_{i=1}^{r} (e_S^i, e_C^i) - \sum_{i=1}^{r-1} (e_S^{i+1}, e_C^i) - (e_S^1, 0) - (0, e_C^r) \\
&= \sum_{i=1}^{r} (e_S^i, 0) + \sum_{i=1}^{r} (0, e_C^i) - \sum_{i=1}^{r-1} (e_S^{i+1}, 0) - \sum_{i=1}^{r-1} (0, e_C^i) - (e_S^1, 0) - (0, e_C^r) \\
&= \sum_{i=1}^{r} (e_S^i, 0) - \sum_{i=1}^{r-1} (e_S^{i+1}, 0) - (e_S^1, 0) + \sum_{i=1}^{r} (0, e_C^i) - \sum_{i=1}^{r-1} (0, e_C^i) - (0, e_C^r) \\
&= \sum_{i=1}^{r} (e_S^i, 0) - \sum_{i=1}^{r} (e_S^i, 0) + \sum_{i=1}^{r} (0, e_C^i) - \sum_{i=1}^{r} (0, e_C^i) = 0.
\end{aligned}
$$

Proof of Lemma 4.2.1

- We have shown that the corresponding column vectors are not linear independent and therefore the considered solution x is not a basic feasible solution
- This is a contradiction to our assumption that x is a basic feasible solution
- Hence, there is no cycle in the corresponding bipartite graph

Proof of Lemma 4.2.1

- Now, we consider the opposite direction and assume that x is a solution with a corresponding bipartite acyclic graph
- We assume that the corresponding columns are not linear independent
- We denote $\{v_1, \ldots, v_k\}$ as the smallest linear dependent subset of the set of all columns belonging to solution x
- Then, it holds that $\exists \lambda_1, \ldots, \lambda_k \neq 0$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ fulfilling $\sum_{i=1}^{k} \lambda_k \cdot v_i = 0$
- Hence, we can construct (by renumbering the respective indices of the columns, the suppliers and the customers) the following sequence:

$$\underbrace{S_1 \rightarrow \rlap{$\overbrace{}^{v_2}$}C_1}_{v_1} \rightarrow \underbrace{S_2 \rightarrow C_2}_{v_3} \rightarrow \ldots \rightarrow \underbrace{S_r \rightarrow \rlap{$\overbrace{\phantom{C_r \rightarrow S_{r+1}}}^{v_k}$}C_r}_{v_{k-1}} \rightarrow S_{r+1}$$

Proof of Lemma 4.2.1

- Note that the latter results from the fact that the set $\{v_1, \dots, v_k\}$ cannot be reduced while each entry represents a specific combination of two unit vectors $e_S^j \in \mathbb{R}^m$, $e_C^i \in \mathbb{R}^n$
- Moreover, as the linear combination results to the zero vector each of the partial unit vectors occurs at least twice
- But, due to the minimality of the set $\{v_1, \dots, v_k\}$, we know that each partial unit vector occurs exactly twice in this set
- As each of these partial vectors occurs twice, this also applies to each supplier and customer in the considered part of the solution x
- Consequently, we can renumber these suppliers and customers accordingly and obtain the aforementioned sequence with $S_1 = S_{r+1}$

$$\underbrace{S_1 \to C_1}_{v_1} \overbrace{\to}^{v_2} \underbrace{S_2 \to C_2}_{v_3} \to \dots \to \underbrace{S_r \to C_r}_{v_{k-1}} \overbrace{\to S_{r+1}}^{v_k} = S_1$$

Proof of Lemma 4.2.1

- However, such a cycle is a contradiction to the assumption that x is a solution with a corresponding bipartite acyclic graph
- Hence, the corresponding columns are linear independent and by applying Definition 1.3.12 and the Remark 4.2.2 (to obtain exactly $n + m - 1$ columns and therefore an invertible matrix A_B), we know that x is a basic feasible solution
- This completes the proof

Considering the reduced costs

- In what follows, we consider a basic feasible solution x of a given Transportation problem
- We want to calculate the reduced without calculating the inverted matrix A_B for a given basis B
- This becomes possible by making use of the following Lemma

Considering the reduced costs

4.2.3 **Lemma**

We consider a basic solution x of the Transportation problem. Moreover, let $\pi^T = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m)$ *and* $\tilde{\pi}^T = \left(\tilde{\alpha}_1, \dots, \tilde{\alpha}_m, \tilde{\beta}_1, \dots, \tilde{\beta}_n\right)$ *be vectors fulfilling the restriction* $\alpha_i + \beta_j = \tilde{\alpha}_i + \tilde{\beta}_j = c_{i,j}$ *for all* $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, n\}$ *with* $x_{i,j}$ *being a basic variable.*

Then, we have $\alpha_i + \beta_j = \tilde{\alpha}_i + \tilde{\beta}_j$ *for all* $i \in \{1, \dots, m\}$ *and* $j \in \{1, \dots, n\}$*.*

Proof of Lemma 4.2.3

- We consider the non-basic variable $x_{k,p}$ of the given basic solution x
- Clearly, the corresponding acyclic bipartite graph consists of $n+m$ vertices and $n+m-1$ edges (see Remark 4.2.2)
- We insert the edge $\left(S_k, C_p\right)$
- This insertion leads to a single cycle as the graph possesses now $n+m$ vertices and $n+m$ edges
- We renumber customers and suppliers such that $k=p=1$ and the single cycle obtains the following structure

$$\boxed{(S_1, C_1), (C_1, S_2), \ldots, (S_r, C_r), (C_r, S_1)}$$

Proof of Lemma 4.2.3

- As all variables $x_{i,i}$ with $i \in \{2, \dots, r\}$ and $x_{i+1,i}$ with $i \in \{1, \dots, r-1\}$ are basic variables
- Therefore, for these tuples i, j we have $\alpha_i + \beta_j = \tilde{\alpha}_i + \tilde{\beta}_j = c_{i,j}$
- We calculate and obtain

$$\begin{aligned}
&(\alpha_2+\beta_1)-(\alpha_2+\beta_2)+(\alpha_3+\beta_2)-\ldots+(\alpha_{i+1}+\beta_i)-(\alpha_i+\beta_i)+\ldots-(\alpha_r+\beta_r)+(\alpha_1+\beta_r)\\
&= c_{2,1} \quad - c_{2,2} \quad + c_{3,2} \quad - \ldots + c_{i+1,i} \quad - c_{i,i} \quad + \ldots - c_{r,r} \quad + c_{1,r} =: c\\
\Leftrightarrow\ & \sum_{i=1}^{r-1}(\alpha_{i+1}+\beta_i)-\sum_{j=2}^{r}(\alpha_j+\beta_j)+(\alpha_1+\beta_r)=c\\
\Leftrightarrow\ & (\sum_{i=2}^{r}\alpha_i)-(\sum_{j=2}^{r}\alpha_j)+(\sum_{i=1}^{r-1}\beta_i)-(\sum_{j=1}^{r-1}\beta_j)+\beta_1-\beta_r+\alpha_1+\beta_r=c\\
\Leftrightarrow\ & \alpha_1+\beta_1=c
\end{aligned}$$

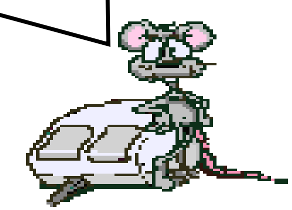
Proof of Lemma 4.2.3

- Thus, for an arbitrary non-basic variable $x_{k,p}$ it holds that $\alpha_k + \beta_p = \tilde{\alpha}_k + \tilde{\beta}_p = c$
- This implies $\alpha_k + \beta_p = \tilde{\alpha}_k + \tilde{\beta}_p$ and completes the proof

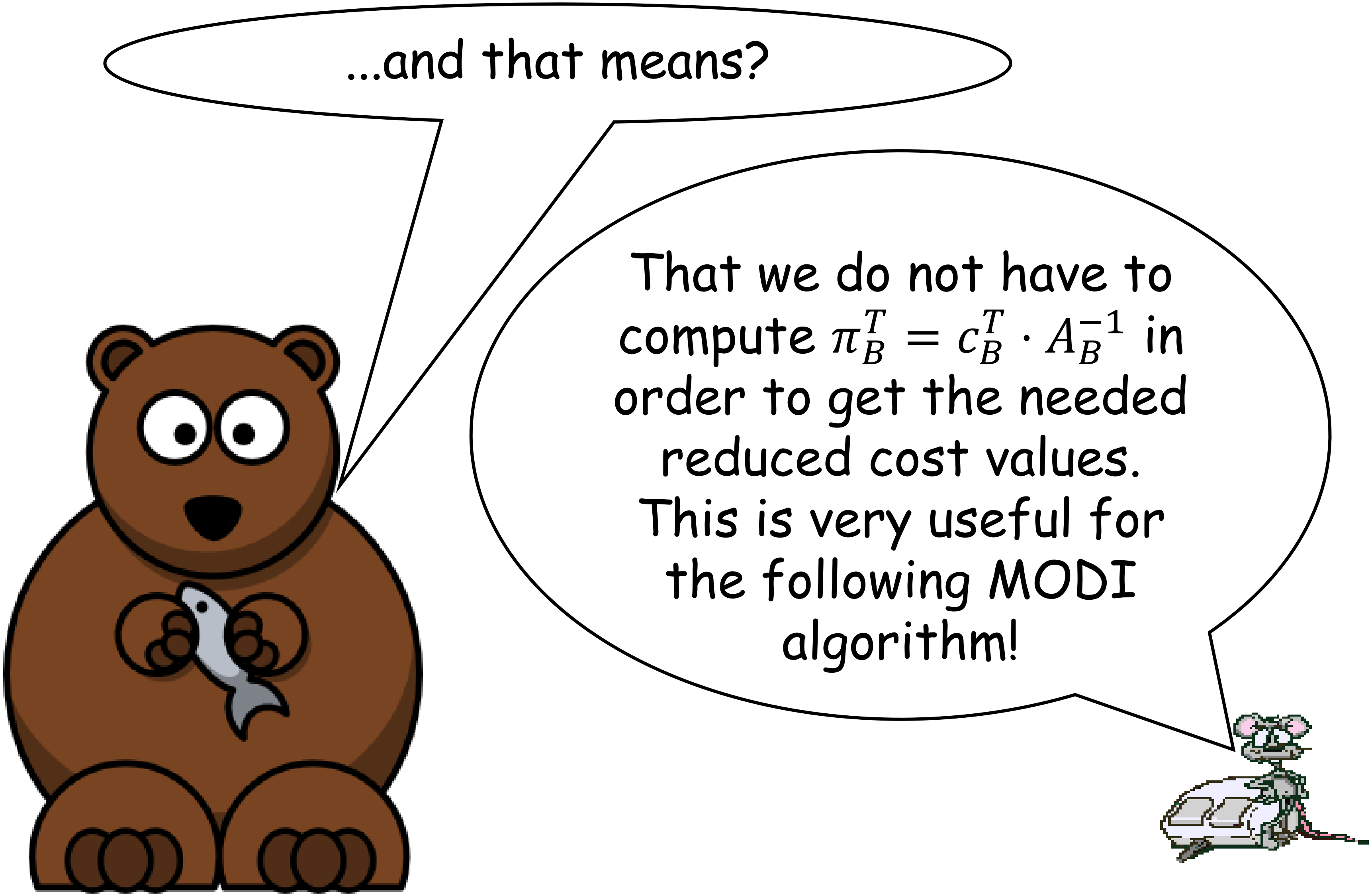
And now?

Nice theory….
But, what is the message? I don't get it!

We can obtain the reduced costs by an arbitrary dual solution that fulfills the criterion $\alpha_i + \beta_j = c_{i,j}$ for all basic variables $x_{i,j}$



And now?


...and that means?
That we do not have to compute $\pi_B^T = c_B^T \cdot A_B^{-1}$ in order to get the needed reduced cost values. This is very useful for the following MODI algorithm!

Cognition

- The dual solution $\pi_B^T = c_B^T \cdot A_B^{-1} = (\alpha_1^B, \dots, \alpha_m^B, \beta_1^B, \dots, \beta_n^B)$ for a given basis B satisfies the property of Lemma 4.2.3
- The reduced costs of the current basic feasible solution x can be calculated by $\bar{c}_{i,j} = c_{i,j} - \left(\alpha_i^B + \beta_j^B\right)$
- However, due to Lemma 4.2.3, we can calculate the reduced costs by using an arbitrary dual solution $\pi^T = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$ satisfying $\alpha_i + \beta_j = c_{i,j}$ for all basic variables $x_{i,j}$

Generating an initial solution

- In the following, we make use of the well-known *Northwest Corner Method*
- It generates a basic feasible solution by conducting the following steps
  1. Start with the northwest corner cell of the matrix.
  2. Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.
  3. Cross out the row or column with zero supply or demand in order to indicate that no further assignment can be made in that row or column. If both, a row and a column, are set to zero simultaneously, cross out only one, and leave a zero supply or demand in the uncrossed row or column.
  4. If exactly one row or column is left out uncrossed, stop. Otherwise, move to the cell to the right if a column has just crossed, or move below if it was a row. Proceed with step 2.

Northwest Corner Method

4.2.4 **Lemma**

The application of the Northwest Corner Method results in a basic feasible solution

Proof of Lemma 4.2.4

- We prove this Lemma by induction over the sum k of the number of suppliers n and the number of customers m, i.e., for $k = n + m$
- We commence this induction with $k = n + m = 2$
  - In this case there is only one edge in the corresponding graph and two nodes
  - This graph is obviously acyclic and the generated solution is a basic feasible solution

Proof of Lemma 4.2.4

- We consider the case $k = n + m > 2$
  - We assume, that the statement holds for all values less than $k = n + m$
  - Consequently, we can apply the assumption of induction if we abstain from the last step of the Northwest Corner Method, as during this step one additional customer or supplier (in what follows denoted by a) is integrated by a transport for the first time
  - Hence, potentially after reducing the demand or supply value of the customer or supplier (note that this is not a) that would remain unsatisfied without the last step, we obtain – by assumption – a feasible basic solution after excluding a
  - For this solution we can generate the corresponding acyclic bipartite graph

Proof of Lemma 4.2.4

  - After that, we modify this graph by inserting a with an additional edge defined by the last step
  - This final step does not produce a cycle and we know by applying Lemma 4.2.1 that the produced solution is a basic feasible solution
- This completes the proof

The MODI algorithm – basic loop

- In what follows, we shall introduce the MODI algorithm and illustrate its application
- Its main loop looks like
- While negative reduced cost entries exist,
  - Determine the smallest reduced cost entry (i, j)
  - Insert the corresponding x-variable into the basis
    - I.e., we have to find a closed loop between the current basis members
    - Then the maximum amount is transferred along this cyclical path
    - Consequently, one element leaves the basis while (i, j) enters it
    - Correct the dual variables accordingly
- Optimal solution found

MODI Algorithm I

1. Find a feasible initial solution to the TPP
2. Determine a dual solution:
   - Set an arbitrary dual variable to ZERO
   - Calculate $\alpha_i = c_{ij} - \beta_j$ for a given β_j or $\beta_j = c_{ij} - \alpha_i$ for a given α_i
   - Use only those cost coefficients for the calculation, where the corresponding primal variable is a basis variable at that time.
3. Calculate the reduced costs $\bar{c}_{ij}$ for all non-basic variables by $\bar{c}_{ij} = c_{ij} - \alpha_i - \beta_j$
4. If $\bar{c}_{ij} \geq 0 \; \forall i, j$, then terminate since the optimal solution is found
5. Otherwise, conduct a basis change (see next slide)

MODI Algorithm II

5. Conduct a basis change
   - Choose the smallest reduced costs $\bar{c}_{pq} = \min\left\{\bar{c}_{ij} \middle| \bar{c}_{ij} < 0 \wedge \forall i, j\right\}$
   - Find a closed loop of basic variables that includes x_{pq}
   - Label x_{pq} with $+\Delta$ and label the remaining basic variables in the circle alternately with $-\Delta$ and $+\Delta$
   - Determine an upper bound $x_{ab} = \min\left\{x_{ij} \middle| (i, j) \text{ is a member of the closed loop and is labeled with } -\Delta\right\}$
   - x_{pq} enters the basis and x_{ab} becomes a non-basic variable
   - Calculate new values for all basic variables in the closed loop according to the labels $x_{ij} := x_{ij} \pm \Delta$
   - Calculate the objective function value $Z := Z + \bar{c}_{pq} \cdot x_{pq}$
   - Go to step 2.

The MODI Algorithm – Example

- We consider the following constellation

$$a = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}; b = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 3 \end{pmatrix}; c = \begin{pmatrix} 3 & 3 & 1 & 2 \\ 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 3 \end{pmatrix}$$

Example – Initial solution

$$c = \left|\begin{array}{ccccc} 2 & 3 & 6 & 3 & \\ (0) & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 3 & 6 & 3 & \\ 2 & (0) & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 2 & 6 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & (0) & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 6 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & (0) & 0 & 3 \\ 0 & 0 & 0 & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 3 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & (0) & 0 & 6 \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 0 & 3 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & (0) & 3 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} 0 & 0 & 0 & 0 & \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right| \Rightarrow x = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}$$

The dual variables

$$x=\begin{pmatrix}2&1&0&0\\0&2&3&0\\0&0&3&3\end{pmatrix},c=\begin{pmatrix}3&3&&\\&2&2&\\&&6&3\end{pmatrix}$$

We commence with $\alpha_1=0$. Thus, we get

$$\left|\begin{array}{ccccc}\beta_1=? & \beta_2=? & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{array}\right| \Rightarrow \left|\begin{array}{ccccc}\beta_1=3 & \beta_2=? & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{array}\right| \Rightarrow$$

$$\left|\begin{array}{ccccc}\beta_1=3 & \beta_2=3 & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=?\\ & & 6 & 3 & \alpha_3=?\end{array}\right| \Rightarrow \left|\begin{array}{ccccc}\beta_1=3 & \beta_2=3 & \beta_3=? & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0\\ & 2 & 2 & & \alpha_2=-1\\ & & 6 & 3 & \alpha_3=?\end{array}\right|$$

The dual variables

$$\left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=? \end{array}\right| \Rightarrow \left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=? & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=3 \end{array}\right|$$

$$\Rightarrow \left|\begin{array}{ccccc} \beta_1=3 & \beta_2=3 & \beta_3=3 & \beta_4=0 & \\ 3 & 3 & & & \alpha_1=0 \\ & 2 & 2 & & \alpha_2=-1 \\ & & 6 & 3 & \alpha_3=3 \end{array}\right|$$

The resulting reduced costs

- Eventually, we obtain the tableau

$$\begin{vmatrix} \beta_1 = 3 & \beta_2 = 3 & \beta_3 = 3 & \beta_4 = 0 & \\ 0 & 0 & -2 & 2 & \alpha_1 = 0 \\ -1 & 0 & 0 & 4 & \alpha_2 = -1 \\ -2 & -1 & 0 & 0 & \alpha_3 = 3 \end{vmatrix}$$

The Tableau – Step 1

- Consider the initial solution

$$
\left|\begin{array}{lcccccc}
\beta = & 3 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & (2)^0 & (1)^0 & 0^{-2} & 0^{2} & 3 & 0 \\
 & 0^{-1} & (2)^0 & (3)^0 & 0^{4} & 5 & -1 \\
 & 0^{-2} & 0^{-1} & (3)^0 & (3)^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right| \Rightarrow Z = 6+3+4+6+18+9 = 46
$$

The Tableau – Step 2

$$
\left|\begin{array}{lcccccc}
\beta = & 3 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 2^0 & 1^0 & 0^{-2} & 0^2 & 3 & 0 \\
 & 0^{-1} & 2^0 & 3^0 & 0^4 & 5 & -1 \\
 & \left[0^{-2}\right] & 0^{-1} & 3^0 & 3^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 3 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & \left(2^0\right) & \left[1^0\right] & 0^{-2} & 0^2 & 3 & 0 \\
 & 0^{-1} & \left(2^0\right) & \left[3^0\right] & 0^4 & 5 & -1 \\
 & \left[0^{-2}\right] & 0^{-1} & \left(3^0\right) & 3^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$$
\left|\begin{array}{lcccccc}
\beta = & (1) & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^2 & 3^0 & 0^{-2} & 0^2 & 3 & 0 \\
 & 0^1 & 0^0 & 5^0 & 0^4 & 5 & -1 \\
 & \left[2^0\right] & 0^{-1} & 1^0 & 3^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
\Rightarrow
\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^2 & (3)^0 & 0^{-2} & 0^2 & 3 & 0 \\
 & 0^1 & (0)^0 & (5)^0 & 0^4 & 5 & -1 \\
 & (2)^0 & 0^{-1} & (1)^0 & (3)^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|
$$

$\Rightarrow Z = 8 + 9 + 0 + 10 + 6 + 9 = 42$, i.e., reduction by $2 \cdot 2$

The Tableau – Step 3

$$\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & 3^{0} & \left[0^{-2}\right] & 0^{2} & 3 & 0 \\
 & 0^{1} & 0^{0} & 5^{0} & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right| \Rightarrow \left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & \left(3^{0}\right) & \left[0^{-2}\right] & 0^{2} & 3 & 0 \\
 & 0^{1} & \left[0^{0}\right] & \left(5^{0}\right) & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|$$

$$\Rightarrow \left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{2} & 0^{0} & \left[3^{-2}\right] & 0^{2} & 3 & (0) \\
 & 0^{1} & 3^{0} & 2^{0} & 0^{4} & 5 & -1 \\
 & 2^{0} & 0^{-1} & 1^{0} & 3^{0} & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right| \Rightarrow \left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^{4} & 0^{2} & \left(3^{0}\right) & 0^{4} & 3 & -2 \\
 & 0^{1} & \left(3^{0}\right) & \left(2^{0}\right) & 0^{4} & 5 & -1 \\
 & \left(2^{0}\right) & 0^{-1} & \left(1^{0}\right) & \left(3^{0}\right) & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|$$

$\Rightarrow Z = 8+6+3+4+6+9 = 36$, i.e., reduction by $3 \cdot 2$

The Tableau – Step 4

$$\left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^4 & 0^2 & (3^0) & 0^4 & 3 & -2 \\
 & 0^1 & (3^0) & (2^0) & 0^4 & 5 & -1 \\
 & (2^0) & [0^{-1}] & (1^0) & (3^0) & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right| \Rightarrow \left|\begin{array}{lcccccc}
\beta = & 1 & 3 & 3 & 0 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^4 & 0^2 & 3^0 & 0^4 & 3 & -2 \\
 & 0^1 & (3^0) & [2^0] & 0^4 & 5 & -1 \\
 & 2^0 & [0^{-1}] & (1^0) & 3^0 & 6 & 3 \\
 & & & & & a & \alpha
\end{array}\right|$$

$$\Rightarrow \left|\begin{array}{lcccccc}
\beta = & (1) & 3 & 3 & (0) & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^4 & 0^2 & 3^0 & 0^4 & 3 & -2 \\
 & 0^1 & (2^0) & [3^0] & 0^4 & 5 & -1 \\
 & 2^0 & [1^{-1}] & (0^0) & 3^0 & 6 & (3) \\
 & & & & & a & \alpha
\end{array}\right| \Rightarrow \left|\begin{array}{lcccccc}
\beta = & 2 & 3 & 3 & 1 & & \\
b = & 2 & 3 & 6 & 3 & & \\
 & 0^3 & 0^2 & (3^0) & 0^3 & 3 & -2 \\
 & 0^0 & (2^0) & (3^0) & 0^3 & 5 & -1 \\
 & (2^0) & (1^0) & 0^1 & (3^0) & 6 & 2 \\
 & & & & & a & \alpha
\end{array}\right|$$

$\Rightarrow Z = 8+4+5+3+6+9 = 35$, i.e., reduction by $1 \cdot 1$

Optimal solution since reduced costs are non-negative!