2 Duality

Papadimitriou and Steiglitz (1982) (p.67):

It would be useful enough if the Simplex Algorithm was all that was provided by the Linear Program Research.

"But there are also many interesting theoretical aspects to the subject, especially relating to combinatorial problems. All of these are related in one way or another to the idea of duality..."

- In what follows, we introduce the **dual of an LP**
- In that coherence, the **original program is denoted as the primal problem**
- By a simultaneous consideration of both programs, it is possible to obtain significant insights into the problem structure of a given instance

2.0 Motivation – Upper bounding

- If we consider a maximization LP as introduced above, we may ask for a bound on the objective function value, i.e., a bound that cannot be exceeded by a feasible solution of the problem
- This will be addressed by reference to a simple example
- Thus, consider the following problem

A simple problem

- Consider the following LP

$$
\begin{aligned}
&\text{Maximize } 4\cdot x_1 + x_2 + 5\cdot x_3 + 3\cdot x_4\\
&\text{s.t.}\\
&x_1 - x_2 - x_3 + 3\cdot x_4 \le 1\\
&5\cdot x_1 + x_2 + 3\cdot x_3 + 8\cdot x_4 \le 55\\
&-x_1 + 2\cdot x_2 + 3\cdot x_3 - 5\cdot x_4 \le 3\\
&x_1, x_2, x_3, x_4 \ge 0
\end{aligned}
$$

Upper bound consideration

- We can state that for each feasible solution x it holds:

$$\begin{aligned}&(x_1 - x_2 - x_3 + 3\cdot x_4)\cdot \pi_1\\ &+(5\cdot x_1 + x_2 + 3\cdot x_3 + 8\cdot x_4)\cdot \pi_2\\ &+(-x_1 + 2\cdot x_2 + 3\cdot x_3 - 5\cdot x_4)\cdot \pi_3\\ &=(\pi_1 + 5\cdot\pi_2 - \pi_3)\cdot x_1 + (-\pi_1 + \pi_2 - 2\cdot\pi_3)\cdot x_2\\ &+(-\pi_1 + 3\cdot\pi_2 + 3\cdot\pi_3)\cdot x_3 + (3\cdot\pi_1 + 8\cdot\pi_2 - 5\cdot\pi_3)\cdot x_4\\ &\leq 1\cdot\pi_1 + 55\cdot\pi_2 + 3\cdot\pi_3,\ \text{with: } x_1,x_2,x_3,x_4,\pi_1,\pi_2,\pi_3 \geq 0\end{aligned}$$

- Consequently, we are able to provide an upper bound on the objective function by the following Linear Program

Generating an upper bound

- Thus, we can state

$$\begin{aligned}&1\cdot\pi_1 + 55\cdot\pi_2 + 3\cdot\pi_3\\ &\text{s.t.}\\ &\pi_1 + 5\cdot\pi_2 - \pi_3 \geq 4\\ &-\pi_1 + \pi_2 + 2\cdot\pi_3 \geq 1\\ &-\pi_1 + 3\cdot\pi_2 + 3\cdot\pi_3 \geq 5\\ &3\cdot\pi_1 + 8\cdot\pi_2 - 5\cdot\pi_3 \geq 3,\ \text{with } \pi_1,\pi_2,\pi_3 \geq 0\end{aligned}$$

- Since we are interested in finding the minimum upper bound, we define the following minimization problem

The corresponding (dual) problem

- The following problem is denoted as the dual problem

$$\begin{aligned}&\text{Minimize } 1\cdot\pi_1 + 55\cdot\pi_2 + 3\cdot\pi_3\\ &\text{s.t.}\\ &\pi_1 + 5\cdot\pi_2 - \pi_3 \geq 4\\ &-\pi_1 + \pi_2 + 2\cdot\pi_3 \geq 1\\ &-\pi_1 + 3\cdot\pi_2 + 3\cdot\pi_3 \geq 5\\ &3\cdot\pi_1 + 8\cdot\pi_2 - 5\cdot\pi_3 \geq 3,\ \text{with } \pi_1,\pi_2,\pi_3 \geq 0\end{aligned}$$

2.1 The dual problem

- In general, the dual of the problem

$$\text{Maximize } \sum_{j=1}^{n} c_j \cdot x_j$$

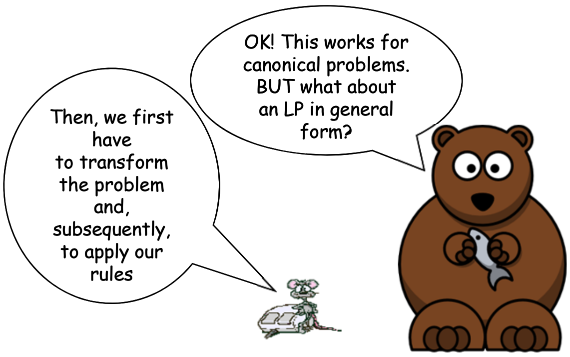
$$\text{subject to } \sum_{j=1}^{n} a_{i,j} \cdot x_j \leq b_i, \forall i \in \{1,\dots,m\}, x_j \geq 0, \forall j \in \{1,\dots,n\}$$

- is defined by

$$\text{Minimize } \sum_{i=1}^{m} b_i \cdot \pi_i$$

$$\text{subject to } \sum_{i=1}^{m} a_{i,j} \cdot \pi_i \geq c_j, \forall j \in \{1,\dots,n\}, \pi_i \geq 0, \forall i \in \{1,\dots,m\}$$

and in general?



For instance: Equation

$$\text{Maximize } \sum_{j=1}^{n} c_j \cdot x_j$$

$$\text{subject to } A \cdot x = b, x \geq 0$$

- Is equivalent to

$$\text{Maximize } c^T \cdot x \text{ subject to } A \cdot x \leq b \wedge -A \cdot x \leq -b, x \geq 0$$

$$\Leftrightarrow$$

$$\text{Maximize } c^T \cdot x \text{ subject to } \begin{pmatrix} A \\ -A \end{pmatrix} \cdot x \leq \begin{pmatrix} b \\ -b \end{pmatrix}, x \geq 0$$

Thus, the dual is

Minimize $b^T \cdot \pi^1 - b^T \cdot \pi^2$ subject to $\begin{pmatrix} A^T & -A^T \end{pmatrix} \cdot \begin{pmatrix} \pi^1 & \pi^2 \end{pmatrix} \geq c$,
with $\pi^1, \pi^2 \geq 0$
$\Leftrightarrow$
Minimize $b^T \cdot \pi^1 - b^T \cdot \pi^2$ subject to $A^T \cdot \pi^1 - A^T \cdot \pi^2 \geq c$,
with $\pi^1, \pi^2 \geq 0$

- Thus, we can interpret the both π-vectors as positive and negative components of a free variable π
- Consequently, we can derive the following dual program

The dual program

Minimize $b^T \cdot \pi$ subject to $A^T \cdot \pi \geq c$, with π free

By doing so, we transform the primal problem

$(LP)_G$
Minimize $c^T \cdot x$ subject to $x_{N^+} \geq 0 \wedge x_{N^-} \leq 0 \wedge x_{N^0}$ free
and $A_{M^+} \cdot x \geq b_{M^+} \wedge A_{M^0} \cdot x = b_{M^0} \wedge A_{M^-} \cdot x \leq b_{M^-}$
We separate matrix A as follows

$$A = \begin{pmatrix} A_{M^+} \\ A_{M^0} \\ A_{M^-} \end{pmatrix} = \begin{pmatrix} A^{N^+} & A^{N^0} & A^{N^-} \end{pmatrix} = \begin{pmatrix} A_{M^+}^{N^+} & A_{M^+}^{N^0} & A_{M^+}^{N^-} \\ A_{M^0}^{N^+} & A_{M^0}^{N^0} & A_{M^0}^{N^-} \\ A_{M^-}^{N^+} & A_{M^-}^{N^0} & A_{M^-}^{N^-} \end{pmatrix},$$

$$b = \begin{pmatrix} b_{M^+} \\ b_{M^0} \\ b_{M^-} \end{pmatrix} \wedge c = \begin{pmatrix} c^{N^+} \\ c^{N^0} \\ c^{N^-} \end{pmatrix}$$

Transforming in standard form

$$\tilde{A} = \begin{pmatrix} A_{M^+}^{N^+} & A_{M^+}^{N^0} & -A_{M^+}^{N^0} & -A_{M^+}^{N^-} & -E_{M^+} & 0 \\ A_{M^0}^{N^+} & A_{M^0}^{N^0} & -A_{M^0}^{N^0} & -A_{M^0}^{N^-} & 0 & 0 \\ A_{M^-}^{N^+} & A_{M^-}^{N^0} & -A_{M^-}^{N^0} & -A_{M^-}^{N^-} & 0 & E_{M^-} \end{pmatrix}$$

$$\tilde{x} = \begin{pmatrix} x_{N^+} \\ x_{N^0}^{+} \\ x_{N^0}^{-} \\ x_{N^-} \\ x_{M^+}^{ov} \\ x_{M^-}^{sl} \end{pmatrix} \geq 0 \wedge \tilde{c} = \begin{pmatrix} c_{N^+} \\ c_{N^0} \\ -c_{N^0} \\ -c_{N^-} \\ 0 \\ 0 \end{pmatrix} \wedge \tilde{b} = \begin{pmatrix} b_{M^+} \\ b_{M^0} \\ b_{M^-} \end{pmatrix}$$

and obtaining the dual

$$\text{Maximize } b^T \cdot \pi = \begin{pmatrix} b_{M^+} & b_{M^0} & b_{M^-} \end{pmatrix} \cdot \begin{pmatrix} \pi_{M^+} \\ \pi_{M^0} \\ \pi_{M^-} \end{pmatrix}$$

$$A^T \cdot \pi = \begin{pmatrix} \left(A_{M^+}^{N^+}\right)^T & \left(A_{M^0}^{N^+}\right)^T & \left(A_{M^-}^{N^+}\right)^T \\ \left(A_{M^+}^{N^0}\right)^T & \left(A_{M^0}^{N^0}\right)^T & \left(A_{M^-}^{N^0}\right)^T \\ \left(-A_{M^+}^{N^0}\right)^T & \left(-A_{M^0}^{N^0}\right)^T & \left(-A_{M^-}^{N^0}\right)^T \\ \left(-A_{M^+}^{N^-}\right)^T & \left(-A_{M^0}^{N^-}\right)^T & \left(-A_{M^-}^{N^-}\right)^T \\ 0 & 0 & E_{M^-} \\ -E_{M^+} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \pi_{M^+} \\ \pi_{M^0} \\ \pi_{M^-} \end{pmatrix} \leq \begin{pmatrix} c^{N^+} \\ c^{N^0} \\ -c^{N^0} \\ -c^{N^-} \\ 0^{M^-} \\ 0^{M^+} \end{pmatrix}$$

And therefore, we obtain

$$\text{Maximize } b^T \cdot \pi = \begin{pmatrix} b_{M^+} & b_{M^0} & b_{M^-} \end{pmatrix} \cdot \begin{pmatrix} \pi_{M^+} \\ \pi_{M^0} \\ \pi_{M^-} \end{pmatrix}$$

$$\left(A^{N^+}\right)^T \cdot \pi \leq c^{N^+} \wedge \left(A^{N^0}\right)^T \cdot \pi \leq c^{N^0} \wedge \left(-A^{N^0}\right)^T \cdot \pi \leq -c^{N^0} \wedge$$
$$\left(-A^{N^-}\right)^T \cdot \pi \leq -c^{N^-} \wedge \pi_{M^-} \leq 0 \wedge -\pi_{M^+} \leq 0$$

$$\Leftrightarrow \text{Maximize } b^T \cdot \pi = \begin{pmatrix} b_{M^+} & b_{M^0} & b_{M^-} \end{pmatrix} \cdot \begin{pmatrix} \pi_{M^+} \\ \pi_{M^0} \\ \pi_{M^-} \end{pmatrix}$$

$$\left(A^{N^+}\right)^T \cdot \pi \leq c^{N^+} \wedge \left(A^{N^0}\right)^T \cdot \pi \leq c^{N^0} \wedge \left(A^{N^0}\right)^T \cdot \pi \geq c^{N^0} \wedge$$
$$\left(A^{N^-}\right)^T \cdot \pi \geq c^{N^-} \wedge \pi_{M^-} \leq 0 \wedge \pi_{M^+} \geq 0$$

And finally

$\Leftrightarrow$

$$\text{Maximize } b^T \cdot \pi = \begin{pmatrix} b_{M^+} & b_{M^0} & b_{M^-} \end{pmatrix} \cdot \begin{pmatrix} \pi_{M^+} \\ \pi_{M^0} \\ \pi_{M^-} \end{pmatrix}$$

$$\left(A^{N^+}\right)^T \cdot \pi \le c^{N^+} \wedge \left(A^{N^0}\right)^T \cdot \pi = c^{N^0} \wedge$$
$$\left(A^{N^-}\right)^T \cdot \pi \ge c^{N^-} \wedge \pi_{M^-} \le 0 \wedge \pi_{M^+} \ge 0$$

Direct comparison

Primal $(LP)_G$ in general form	Dual $(LP)_G$ in general form
Minimize $c^T \cdot x$	Maximize $b^T \cdot \pi$
subject to	subject to
$A_{M^+} \cdot x \ge b_{M^+}$	$\pi_{M^+} \ge 0$
$A_{M^0} \cdot x = b_{M^0}$	π_{M^0} free
$A_{M^-} \cdot x \le b_{M^-}$	$\pi_{M^-} \le 0$
$x^{N^+} \ge 0$	$\left(A^{N^+}\right)^T \cdot \pi \le c^{N^+}$
$x^{N^-} \le 0$	$\left(A^{N^-}\right)^T \cdot \pi \ge c^{N^-}$
x^{N^0} free	$\left(A^{N^0}\right)^T \cdot \pi = c^{N^0}$

Example

(P)

$$\text{Minimize } (1,-3,5,2) \cdot x$$

subject to $x_1 \ge 0 \wedge x_2, x_3$ free $\wedge\, x_4 \le 0$

$$\begin{pmatrix} 1 & -3 & 2 & 1 \\ -1 & -1 & 5 & 0 \end{pmatrix} \cdot x \ge \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$
$$\wedge \begin{pmatrix} 5 & 1 & 0 & 0 \end{pmatrix} \cdot x = 6 \wedge \begin{pmatrix} -1 & 0 & 0 & 3 \end{pmatrix} \cdot x \le -10$$

Example – Preparing the problem

$$\begin{pmatrix} 1 & -3 & 2 & 1 \\ -1 & -1 & 5 & 0 \end{pmatrix} \cdot x \geq \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

$$\wedge \begin{pmatrix} 5 & 1 & 0 & 0 \end{pmatrix} \cdot x = 6 \wedge \begin{pmatrix} -1 & 0 & 0 & 3 \end{pmatrix} \cdot x \leq -10$$

$$\Rightarrow A = \begin{pmatrix} 1 & -3 & 2 & 1 \\ -1 & -1 & 5 & 0 \\ 5 & 1 & 0 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & -1 & 5 & -1 \\ -3 & -1 & 1 & 0 \\ 2 & 5 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

Generating the dual

(D)

Maximize $(-1,3,6,-10) \cdot \pi$

subject to

$$\begin{pmatrix} 1 & -1 & 5 & 1 \end{pmatrix} \cdot \pi \leq 1 \wedge \begin{pmatrix} -3 & -1 & 1 & 0 \end{pmatrix} \cdot \pi = -3$$

$$\wedge \begin{pmatrix} 2 & 5 & 0 & 0 \end{pmatrix} \cdot \pi = 5 \wedge \begin{pmatrix} 1 & 0 & 0 & 3 \end{pmatrix} \cdot \pi \geq 2$$

$\pi_1, \pi_2 \geq 0 \wedge \pi_3$ free $\wedge \pi_4 \leq 0$

Comparing the primal and dual objective value

- We consider the problems

  (P): Max $c^T \cdot x$, s.t. $A \cdot x \leq b \wedge x \geq 0$ $\qquad$ (D): Min $b^T \cdot \pi$, s.t. $A^T \cdot \pi \geq c \wedge \pi \geq 0$
- For every feasible primal and dual solution it holds:

$$\sum_{j=1}^{n} c_j \cdot x_j \leq \sum_{i=1}^{m} b_i \cdot \pi_i$$

- This is true since

$$\sum_{j=1}^{n} c_j \cdot x_j \leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{i,j} \cdot \pi_i \right) \cdot x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{i,j} \cdot x_j \right) \cdot \pi_i \leq \sum_{i=1}^{m} b_i \cdot \pi_i$$

- or by matrix transformations

$b^T \cdot \pi = \pi^T \cdot b \geq \pi^T \cdot A \cdot x = A^T \cdot \pi \cdot x \geq c^T \cdot x$,

since π is feasible, i.e., $A^T \cdot \pi \geq c$

Main cognition – Optimality criteria

2.1.1 Lemma

Assume that x and π are feasible solutions for (P) and (D), with

$(P)\,Max\;\; c^T \cdot x, s.t., A \cdot x = b, x \geq 0$

$(D)\,Min\;\; b^T \cdot \pi, s.t., A^T \cdot \pi \geq c, \pi\; free$

Then, it holds:

$b^T \cdot \pi = c^T \cdot x \Leftrightarrow x$ and π are optimal solutions

Proof of Lemma 2.1.1

- We assume that it holds: $b^T \cdot \pi = c^T \cdot x$

  Since we know $b^T \cdot \pi \geq c^T \cdot x, \forall x, \pi$

  x and π are optimal solutions for (P) and (D)

- Other way round, we assume that x and π are optimal solutions for (P) and (D)

  Consider the final tableau that generates x^0 as an optimal bfs. Then, we have a corresponding π_0 with $\pi_0 = c_B^T \cdot A_B^{-1}$ and it holds:

  $c^T \cdot x = c^T \cdot x^0 = c_B^T \cdot x_B^0 = c_B^T \cdot A_B^{-1} \cdot b = \pi_0^T \cdot b$

Conclusion

And since x^0 is optimal, we know $c^T - \pi_0^T \cdot A \leq 0 \Rightarrow$

$A^T \cdot \pi_0 \geq c$

therefore, π_0 is feasible

$\Rightarrow c^T \cdot x = c^T \cdot x_0 = \pi_0^T \cdot b \geq \pi^T \cdot b$

and thus altogether

$\Rightarrow c^T \cdot x \geq \pi^T \cdot b \wedge c^T \cdot x \leq \pi^T \cdot b \Rightarrow c^T \cdot x = \pi^T \cdot b$

Observations:

- Therefore, we know that both solutions (x_0 and π_0) have always identical objective function values during the calculation process of the Simplex Algorithm
- If π_0 becomes finally feasible, optimality is proven

2.1.2 The dual of the dual

- We know, the dual of the problem

$$\text{Maximize} \sum_{j=1}^{n} c_j \cdot x_j$$

$$\text{subject to} \sum_{j=1}^{n} a_{i,j} \cdot x_j \le b_i, \forall i \in \{1,\ldots,m\}, x_j \ge 0, \forall j \in \{1,\ldots,n\}$$

- is defined by

$$\text{Minimize} \sum_{i=1}^{m} b_i \cdot \pi_i$$

$$\text{subject to} \sum_{i=1}^{m} a_{i,j} \cdot \pi_i \ge c_j, \forall j \in \{1,\ldots,n\}, \pi_i \ge 0, \forall i \in \{1,\ldots,m\}$$

The dual of the dual

- And this is equivalent to

$$\text{Maximize} \sum_{i=1}^{m} -b_i \cdot \pi_i$$

$$\text{subject to} \sum_{i=1}^{m} -a_{i,j} \cdot \pi_i \le -c_j, \forall j \in \{1,\ldots,n\}, \pi_i \ge 0, \forall i \in \{1,\ldots,m\}$$

- …and the dual of the dual

$$\text{Minimize} \sum_{j=1}^{n} -c_j \cdot x_j$$

$$\text{subject to} \sum_{j=1}^{n} -a_{i,j} \cdot x_i \ge -b_i, \forall i \in \{1,\ldots,m\}, x_i \ge 0, \forall i \in \{1,\ldots,m\}$$

And returning to the original problem

- And this is equivalent to

$$\text{Maximize} \sum_{j=1}^{n} c_j \cdot x_j$$

$$\text{subject to} \sum_{j=1}^{n} a_{i,j} \cdot x_i \le b_i, \forall i \in \{1,\ldots,m\}, x_i \ge 0, \forall i \in \{1,\ldots,m\}$$

2.2 The possible cases

Result:

One of the following constellations applies for each pair (P) and (D):

1. Both problems have an optimal solution

2. None of them has a feasible solution

3. One has no feasible solution while the other one is unbounded

The cases

- We distinguish if there are feasible solutions for (P) and (D)
- Thus, we get the following resulting constellations

	P not empty	P empty
D not empty	1	3
D empty	3	2

The cases

- **Case 1**: Since both problems are solvable, the objective functions are bounded accordingly. Thus, optimal solutions exist
- **Case 2**: trivial
- **Case 3**: Since an optimal solution for (P) would also provide an optimal solution for (D), we can conclude that (P) is unbounded

- We can easily show that all three cases exist
- This is depicted on the following slides…

Example – Case 1

$$(P)\quad \text{Minimize } (1,1)\cdot x \quad \text{s.t. } x \ge 0 \wedge \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is single solution

$\Rightarrow$

$$(D)\quad \text{Maximize } (0,0)\cdot \pi \quad \text{s.t. } \pi \text{ free} \wedge \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \pi \le \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is optimal solution

Example – Case 2

(P)

$$\text{Minimize } (-1 \quad 0 \quad 0)\cdot x \quad \text{s.t. } x \ge 0 \wedge \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\Rightarrow x_2 = -1 < 0 \Rightarrow$ Not solvable

$\Rightarrow (D)$

$$\text{Maximize } (0,-1)\cdot \pi \quad \text{s.t. } \pi \text{ free} \wedge \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \pi \le \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \pi_1 \le -1 \wedge -\pi_1 \le 0 \Rightarrow \pi_1 \ge 0 \Rightarrow$ Not solvable

Example – Case 3

$$(P)\quad \text{Minimize } (0 \quad 0 \quad 0)\cdot x \quad \text{s.t. } x \ge 0 \wedge \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot x = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$\Rightarrow x_2 = -1 < 0 \Rightarrow$ Not solvable

$$\Rightarrow (D)\quad \text{Maximize } (0,-1)\cdot \pi \quad \text{s.t. } \pi \text{ free} \wedge \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \pi \le \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \pi = (0,-a), a \in IR \wedge a \ge 0$ is a feasible solution

$\Rightarrow$ Unbounded solution space and therefore no optimal solution available

2.3 The Dual Simplex Algorithm

- If no primal solution is available, we may use the Two-Phase Method or make use of the Dual Simplex Algorithm
- This prerequisites, however, the existence of a feasible dual solution
- The Primal Simplex Algorithm ((P) Min-Problem)

$$\boxed{\text{Invariant}: \overline{b} = A_B^{-1} \cdot b \geq 0 \wedge \text{Optimality criterion}: \overline{c} \geq 0}$$

- The Dual Simplex Algorithm ((P) Min-Problem)

$$\boxed{\text{Invariant}: \overline{c} \geq 0 \wedge \text{Optimality criterion}: \overline{b} = A_B^{-1} \cdot b \geq 0}$$

The Dual Simplex Algorithm

1. Transform the primal problem into a canonical form, generate equations and a minimization objective function.
2. Initialization with a solution to the dual problem.
3. If there are *strictly* negative right-hand side coefficients $\overline{b}$, then
   - Pivot row s (largest coefficient rule): $\min\left\{\overline{b}_i < 0 \middle| \forall i = 1,\ldots,m\right\}$.
   - If $\overline{a}_{sj} \geq 0 \; \forall j = 1,\ldots,n$, then terminate since the dual solution space is unbounded, and thus the primal solution space is empty.
   - Pivot column t: $\min\left\{-\overline{c}_j / \overline{a}_{sj} \middle| \forall j = 1,\ldots,n : \overline{a}_{sj} < 0\right\}$ that is an upper bound on the corresponding dual slack variable t.
   - Basis change: x_t enters the basis and $x_{B(s)}$ becomes a non-basic variable. Apply a linear transformation of the constraint equalities by the Gauß-Jordan algorithm to yield a unit vector with $\overline{a}_{st}=1$ at the pivot element (i.e., e^s in column t). Go to step 3.

   Otherwise: Termination. A feasible basic solution to the primal problem is found.

The Dual Simplex - Example

(P)

$$\text{Minimize } (4,2) \cdot x \quad \text{s.t. } x \geq 0 \wedge \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \cdot x \geq \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\text{Minimize } \begin{pmatrix} 4 & 2 & 0 & 0 \end{pmatrix} \cdot x \quad \text{s.t. } x \geq 0 \wedge \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix} \cdot x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\Rightarrow$

$$\begin{array}{c|cccc} 0 & 4 & 2 & 0 & 0 \\ \hline 2 & 1 & 2 & -1 & 0 \\ 1 & 1 & -1 & 0 & -1 \end{array} \Rightarrow \begin{array}{c|cccc} 0 & 4 & 2 & 0 & 0 \\ \hline -2 & -1 & -2 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 \end{array} \Rightarrow x = \begin{pmatrix} 0 & 0 & -2 & -1 \end{pmatrix}^T$$

$\Rightarrow$ Obviously, $x = \begin{pmatrix} 0 & 0 & -2 & -1 \end{pmatrix}^T$ is not feasible!

The dual problem

$$(D)$$

$$\Rightarrow \text{Minimize } \begin{pmatrix}-2 & -1 & 0 & 0\end{pmatrix}\cdot\pi \ \text{ s.t. } \pi \geq 0 \wedge \begin{pmatrix}1 & 1 & 1 & 0\\ 2 & -1 & 0 & 1\end{pmatrix}\cdot\pi = \begin{pmatrix}4\\2\end{pmatrix}$$

$$\begin{array}{c|cccc} 0 & -2 & -1 & 0 & 0\\ \hline 4 & 1 & 1 & 1 & 0\\ 2 & 2 & -1 & 0 & 1\end{array} \Rightarrow \pi = \begin{pmatrix}0 & 0 & 4 & 2\end{pmatrix}^T \ \text{feasible dual solution!}$$

- We have a dual solution directly
- Consequently, we can either conduct the Primal Simplex on the dual tableau or the Dual Simplex on the primal tableau

Dual and Primal Simplex

- The Dual Simplex

$$\begin{array}{c|cccc} 0 & 4 & 2 & 0 & 0\\ \hline -2 & -1 & (-2) & 1 & 0\\ -1 & -1 & 1 & 0 & 1\end{array} \Rightarrow \begin{array}{c|cccc} -2 & 3 & 0 & 1 & 0\\ \hline 1 & 1/2 & 1 & -1/2 & 0\\ -2 & -3/2 & 0 & 1/2 & 1\end{array} \Rightarrow \begin{array}{c|cccc} -2 & 3 & 0 & 1 & 0\\ \hline 1 & 1/2 & 1 & -1/2 & 0\\ -2 & (-3/2) & 0 & 1/2 & 1\end{array}$$

$$\Rightarrow \begin{array}{c|cccc} -6 & 0 & 0 & 2 & 2\\ \hline 1/3 & 0 & 1 & -1/3 & 1/3\\ 4/3 & 1 & 0 & -1/3 & -2/3\end{array} \Rightarrow x = \begin{pmatrix}4/3 & 1/3\end{pmatrix}^T \ \text{feasible and optimal}$$

- The primal applied to the dual problem

$$\begin{array}{c|cccc} 0 & -2 & -1 & 0 & 0\\ \hline 4 & 1 & 1 & 1 & 0\\ 2 & (2) & -1 & 0 & 1\end{array} \Rightarrow \begin{array}{c|cccc} 2 & 0 & -2 & 0 & 1\\ \hline 4 & 1 & 1 & 1 & 0\\ 1 & 1 & -1/2 & 0 & 1/2\end{array} \Rightarrow \begin{array}{c|cccc} 2 & 0 & -2 & 0 & 1\\ \hline 3 & 0 & 3/2 & 1 & -1/2\\ 1 & 1 & -1/2 & 0 & 1/2\end{array}$$

$$\begin{array}{c|cccc} 2 & 0 & -2 & 0 & 1\\ \hline 3 & 0 & (3/2) & 1 & -1/2\\ 1 & 1 & -1/2 & 0 & 1/2\end{array} \Rightarrow \begin{array}{c|cccc} 2 & 0 & -2 & 0 & 1\\ \hline 2 & 0 & 1 & 2/3 & -1/3\\ 1 & 1 & -1/2 & 0 & 1/2\end{array} \Rightarrow \begin{array}{c|cccc} 6 & 0 & 0 & 4/3 & 1/3\\ \hline 2 & 0 & 1 & 2/3 & -1/3\\ 2 & 1 & 0 & 1/3 & 1/3\end{array}$$

$$\Rightarrow \pi^{opt} = (2,2,0,0)^T \Rightarrow x = \begin{pmatrix}4/3 & 1/3\end{pmatrix}^T$$

Pivoting in the Dual Simplex

- Owing to the exchanged invariant and optimality criterion, we have the following modified pivoting rule
- I.e., we always select the subsequent column according to the following rule

$$\text{Select } s \text{ with } \overline{b}_s < 0; \text{Select } t \text{ as } \frac{\overline{c}_t}{-a_s^t} = \min\left\{\frac{\overline{c}_t}{-a_s^j} \mid a_s^j < 0\right\}$$

2.4 Interpreting the dual

- By analyzing the results provided by primal or dual programs, we try to obtain insights into the respective problem structure
- For several examples, there are interesting economical interpretations possible
- Hence, in what follows, we consider several examples in detail …

Production Program Planning

- We make use of altogether four resources 1,2,3, and 4 in order to produce a theoretical production program of two product types A and B
- Each product type (A and B) comes along with individual marginal profits and consumption rates for using the resources
- We measure the profits by monetary units per product unit and the consumption rates by resource units per product unit

The mathematical problem

$$\text{Maximize } 60 \cdot x_A + 100 \cdot x_B, x_A \geq 0 \wedge x_B \geq 0$$

$$\text{s.t. } \frac{1}{10} \cdot x_A + \frac{1}{8} \cdot x_B \leq 150 \wedge \frac{1}{9} \cdot x_A + \frac{1}{10} \cdot x_B \leq 150$$

$$\wedge \frac{1}{8} \cdot x_A + 0 \cdot x_B \leq 150 \wedge 0 \cdot x_A + \frac{1}{6} \cdot x_B \leq 150$$

$$\Rightarrow \begin{pmatrix} \frac{1}{10} & \frac{1}{8} \\ \frac{1}{9} & \frac{1}{10} \\ \frac{1}{8} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} x_A \\ x_B \end{pmatrix} \leq \begin{pmatrix} 150 \\ 150 \\ 150 \\ 150 \end{pmatrix}$$

The dual program

- We obtain

$$\text{Minimize } 150\cdot\pi_1 + 150\cdot\pi_2 + 150\cdot\pi_3 + 150\cdot\pi_4$$

s.t.

$$\begin{pmatrix} \frac{1}{10} & \frac{1}{9} & \frac{1}{8} & 0 \\ \frac{1}{8} & \frac{1}{10} & 0 & \frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} \geq \begin{pmatrix} 60 \\ 100 \end{pmatrix} \wedge \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} \geq 0$$

How to interpret the dual program?

- First, consider the basis units…
- Obviously, the variables are measured in monetary units per resource units
- Let us assume that the predetermined resources are held by a vendor for prices π_1, π_2, π_3, π_4 each
- Thus, the objective function minimizes the procurement of 150 resource units for all resources
- But: Other way round, the vendor of the resources can use the resource units on its own in order to produce the product types A and B
- Thus, the marginal profits of the products are lower bounds for the prices of the resource units

We solve the program by Excel

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Primal and dual solution

- Thus, we obtain the primal solution

$$\boxed{\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 375 \\ 900 \end{pmatrix}}$$

- and the dual

$$\boxed{\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 600 \\ 0 \\ 0 \\ 150 \end{pmatrix}}$$

Interpreting the result

- The price for resource 1 is 600, i.e., this resource is short
- Specifically, we are willing to pay up to 600$ for an additional resource item
- Analogously, resource 4 is short as well and we are willing to pay up to 150$ for each item
- Thus, the values of the optimal solution to the dual problem are usually denoted as **shadow prices**
- In contrast to this, resources 2 and 3 are not short, i.e., we are not willing to pay for their additional availability

Consequences for the primal problem

- With the previous constellation we achieve a profit of 112.500$
- Let us now modify the primal problem accordingly and **extend the resource restriction 1**

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New solution

- Now we get

$$\boxed{\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 385 \\ 900 \end{pmatrix}}$$

- and a total profit of 113.100$, i.e., an increase of 113.100-112.500=600$, just as anticipated

Extending resource 2

- With the previous constellation we achieve a profit of 112.500$
- Let us now modify the primal problem accordingly and extend the resource restriction 2

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Consequences for the primal problem

- With the previous constellation we achieve a profit of 112.500$
- Let us now modify the primal problem accordingly and **extend the resource restriction 4**

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New solution

- Now we get

$$\boxed{\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 367{,}5 \\ 906 \end{pmatrix}}$$

- and a total profit of 112.650\$, i.e., an increase of 112.650\$-112.500\$=150\$, just as anticipated

Example 2 – The Diet Problem

$$(P)$$

$$\text{Minimize } (20{,}30)\cdot\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, x \geq 0 \wedge \begin{pmatrix} 2 & 5 \\ 3 & 4 \\ 2 & 1 \end{pmatrix} \cdot x \geq \begin{pmatrix} 10 \\ 12 \\ 5 \end{pmatrix}$$

$$(D)$$

$$\text{Maximize } (10 \quad 12 \quad 5)\cdot\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}, \pi \geq 0 \wedge \begin{pmatrix} 2 & 3 & 2 \\ 5 & 4 & 1 \end{pmatrix} \cdot \pi \leq \begin{pmatrix} 20 \\ 30 \end{pmatrix}$$

Interpreting the primal problem

- Find an efficient healthy nutrition
- Two kinds of food may be consumed
- A housekeeper has to find a food combination that guarantees a healthy nutrition at the least possible costs
- Lower bounds are defined as minimal amounts of vitamins that have to be consumed in the considered planning horizon

Interpreting the dual problem

- We assume that there is an additional vendor of vitamins
- He or she offers the daily package and wants to maximize the profit
- Obviously, a combination equal to one of the food ingredients has to cost at most the same price. Otherwise, no one would consume it but the food instead
- The variables are the prices per vitamin

Solving the primal problem

- We make use again of the Excel Solver

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Optimal solution to the primal

- We obtain the solution

$$\boxed{\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 2{,}86 \\ 0{,}86 \end{pmatrix}}$$

- Objective function value

$$\boxed{Z(x_A, x_B) = 82{,}85714286}$$

- Vitamin 1 and 2 are consumed just completely
- Vitamin 3 is consumed more than necessary

Solving the dual problem

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Optimal solution to the dual

- We obtain the solution

$$\boxed{\begin{pmatrix}\pi_1\\ \pi_2\\ \pi_3\end{pmatrix} = \begin{pmatrix}1{,}42857143\\ 5{,}71428571\\ 0\end{pmatrix}}$$

- Objective function value

$$\boxed{Z(\pi_1,\pi_2,\pi_3) = 82{,}85714286}$$

Interpreting the result

- Vitamin 3 costs nothing since it is excessively consumed
- If we have to consume one more unit of vitamin 1, the total result is deteriorated by 1.43$
- If we have to consume one more unit of vitamin 2, the total result is deteriorated by 5.71$

Illustration

- Showing the impact of shadow prices at the dual problem

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- As anticipated, our costs increased by 1,42857143 (**just equal to respective shadow price**)

2.5 Farkas' Lemma

- Farkas' Lemma is a fundamental result about vectors in IR^n that in a sense captures the ideas of duality
- It allows to derive the results proven earlier in this course
- Now, we are able to prove Farkas' lemma as a consequence of what we already know about linear programming

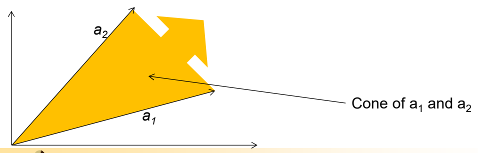
Cones

2.5.1 Definition

Given a set of vectors $a_i \in IR^n$, $i = 1,\dots,m$, the cone generated by the set $\{a_i\}$, denoted by $C(a_i)$, is defined by:

$$C(a_i) = \left\{ x \in IR^n \,\middle|\, x = \sum_{i=1}^{m} \pi_i \cdot a_i, \pi_i \geq 0, i = 1,\dots,m \right\}.$$

2.5.2 Example

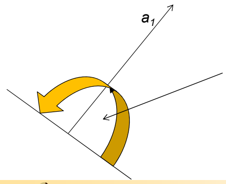


Nonnegative projections

2.5.3 Definition

A vector $c \in IR^n$ has a nonnegative projection on a vector $d \in IR^n$ if it holds that $c^T \cdot d \geq 0$

2.5.4 Example



Farkas' Lemma

2.5.5 Theorem (Farkas' Lemma)

Given a set of vectors $a_i \in IR^n$, $i = 1,...,m$, and another vector $c \in IR^n$, then it holds:

$\forall y \in IR^n : \forall i \in \{1,...,m\} : y^T \cdot a_i \geq 0 \Rightarrow y^T \cdot c \geq 0$

$\Leftrightarrow$

$c \in C(a_1, \quad \ldots \quad ,a_m)$

Proof of Farkas' Lemma

- We start with the second part and assume that

$$c \in C(a_i) \Rightarrow c = \sum_{i=1}^{m} \pi_i \cdot a_i, \text{ with } \pi_i \geq 0$$

- Thus, it holds that

$\exists y \in IR^n : \forall i \in \{1,...,m\} : y^T \cdot a_i \geq 0$ we obtain

$$y^T \cdot c = y^T \cdot \sum_{i=1}^{m} \pi_i \cdot a_i = \sum_{i=1}^{m} \pi_i \cdot \underbrace{y^T \cdot a_i}_{\geq 0} \geq 0, \text{ since } \pi_i \geq 0$$

Proof of Farkas' Lemma

- Now, we assume that the first part holds, i.e., we have

$$\forall y \in IR^n : \forall i \in \{1,...,m\} : y^T \cdot a_i \geq 0 \Rightarrow y^T \cdot c \geq 0$$

- Consider the following Linear Program

$$\begin{aligned} &\min \; c^T \cdot y \\ &\text{s.t. } a_i^T \cdot y \geq 0, \forall i = 1,...,m \\ &y \text{ free} \end{aligned}$$

- This program is obviously feasibly solvable since *y=0* is a feasible solution. It is also bounded since

$$\forall y \in IR^n : \forall i \in \{1,...,m\} : y^T \cdot a_i \geq 0 \Rightarrow y^T \cdot c \geq 0$$

Proof of Farkas' Lemma

- The corresponding dual program is

$$\begin{aligned} &\max \; 0 \\ &\text{s.t. } \pi^T \cdot A_j = c_j, \forall j = 1,...,n \\ &\pi \geq 0 \end{aligned}$$

- Since the primal is solvable and bounded, the dual program is also solvable (see Section 2.2)
- Hence, there exists a vector $\pi \geq 0$ with

$$c = \sum_{i=1}^{m} \pi_i \cdot a_i \text{ and } \pi_i \geq 0$$

Additional literature to Section 2

- Farkas, J. (1902): Theorie der Einfachen Ungleichungen. Journal Reine und Angewandte Mathematik 124 (1902), pp.1-27.

J. von Neumann is credited by D. Gale with being the first to state the duality theorem. Gale (1950) cites the first proof, based on von Neumann's notes, in

- Gale, D.H.; Kuhn, H.W.; Tucker, A.W. (1950): On Symmetric Games in Kuhn, H.W.; Tucker, A.W. (eds.): Contributions to the Theory of Games. Ann. Math Studies, no. 24. Princeton University Press, Princeton, N.J.
- Gale, D. (1960): The Theory of Linear Economic Models McGraw Hill Book Company, New York.